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THE UNSTEADY SLIP-FLOW BOUNDARY LAYER AT HIGH MACH NUMBERS

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "The Unsteady Slip-Flow Boundary Layer at High Mach Numbers" submitted by Dennett Douglas Jaques Netterville in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The transient behavior of the boundary layer on a flat plate following an impulsive acceleration of the surrounding fluid is analysed. The effects of low density are partially accounted for by using boundary conditions which allow a slip velocity and a temperature jump at the wall. The equations governing the boundary layer are rewritten for a coordinate system suggested by the slip boundary conditions, and the resulting equations are solved numerically. Expressions for the shear stress and heat transfer at the wall are developed in terms of the numerical solution, and several solutions are compared with those in the literature.

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LIST OF SYMBOLS

a	velocity of sound
A	a constant defined by equation (A.2)
B	a constant defined by equation (A.21)
C	the Chapman-Rubesin constant, equation (2.20)
C_f	local skin friction coefficient, equation (4.3)
C_p	specific heat at constant pressure
e	proportionality constant appearing in equation (2.43)
f	nondimensional stream function defined by (2.39)
F	stream function f as modified by (3.1)
g	nondimensional local enthalpy ratio h/h_e
G	total enthalpy ratio S as modified by (3.2)
h	local enthalpy
H	total enthalpy
i	one coordinate in a two-dimensional grid (see Chapter III)
I_n	independent variable evaluated at the n^{th} grid point
k	the other coordinate in a two dimensional grid
k	thermal conductivity
k	proportionality constant appearing in equation (C.17)
K_n	local Knudsen number at the wall, $\frac{\lambda_w}{\nu_w/u_w}$
L	a quantity defined by equation (A.5)
m	the exponent in equation (2.43)

LIST OF SYMBOLS (continued)

M	Mach number
Nu	Nusselt number defined by (4.23)
P	static pressure
Pr	Prandtl number
q	energy transfer due to heat conduction, equation (4.11)
Q	total energy transfer, equation (4.10)
R	the gas constant
Re_x	local free stream Reynolds number, $U_\infty x/\nu_\infty$
S	nondimensional total enthalpy ratio, H/H_e
t	the time variable
$\bar{t}$, $\bar{T}$, T	transformed time variables
u	velocity in the x-direction
U	free-stream velocity
v	velocity in the y-direction
V	modified free-stream velocity, equation (2.36)
x	coordinate along the plate
$\bar{x}$, $\bar{X}$, X	transformed plate coordinates
y	coordinate normal to the plate
$\bar{y}$, $\bar{Y}$, Y	transformed normal coordinates

LIST OF SYMBOLS (continued)

α	energy accommodation coefficient, assumed to be unity
α	the exponent appearing in equation (B.8)
β	pressure gradient parameter
γ	ratio of specific heats
ε	Aroesty's small perturbation parameter, equation (C.1)
ζ	space-time variable modified by (3.3)
ζ	Aroesty's transformed x-coordinate, equation (C.4)
η	transformed space variable; dependent on x and y
η_A	Aroesty's transformed space variable; equation (C.3)
θ	temperature
λ	molecular mean free path, equation (B.1)
μ	absolute viscosity
ν	kinematic viscosity
ξ	transformed space variable; dependent only on x
ρ	density
σ	tangential momentum accommodation coefficient, assumed to be unity except for correlation with Berard (σ = 0.6)
σ	slip parameter, equation (1.6)
τ	transformed space and time variable; dependent on x and t
Φ	shear stress $\mu \frac{\partial u}{\partial y}$
ψ	dimensional stream function, equations (2.12) and (2.15)

LIST OF SYMBOLS (continued)

Subscripts

ad	evaluated at the adiabatic condition
B	the Blasius solution (steady state, zero slip)
e	evaluated in the free stream (or edge of boundary layer)
0	free stream stagnation value
p	property of the wall (called "the plate")
ref	reference condition
w	property of gas adjacent to the wall
δ	evaluated at edge of boundary layer
1	initial steady state
2	condition just after the impulsive change in free-stream velocity
3	final steady state

Superscripts

*	differentiation with respect to ξ
'	differentiation with respect to η
.	differentiation with respect to τ

CHAPTER I

INTRODUCTION

1.1 Statement of Objectives

There is at present an increasing interest in high-altitude flight where the air density is very low and the continuum regime flow conditions fail to apply. Thus a theory which attempts to describe the behavior of high-altitude high speed machines should take into account as many of the known characteristics of low-density gases as possible. Such characteristics include dissociation and ionization at high temperatures, free-molecule interactions, and velocity and temperature discontinuities at solid boundaries. When evaluating the forces acting on an object subjected to a transient high-speed flow, viscous interactions have to be analysed. Therefore an investigation into the characteristics of a time-dependent boundary layer in a rarefied gas is an attractive and interesting problem. A sharp-leading-edge body will be considered in this investigation.

The problem to be discussed here is one in which the transient behavior of the boundary layer on a flat plate following an impulsive acceleration of the surrounding fluid is analysed. The hypersonic assumption [1]

$$\frac{\partial U_e}{\partial t} = \frac{\partial h_e}{\partial t} = \frac{\partial H_e}{\partial t} = \frac{\partial P_e}{\partial t} = 0 \ , \tag{1.1}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

where

$$H = h + \frac{u^2+v^2}{2} \simeq h + \frac{u^2}{2} ,$$

is used throughout the analysis. These relations are based on the assumption that the free stream adjusts itself instantaneously to the new conditions, whereas the boundary layer itself is non-stationary (a similar case is mentioned by Lagerstrom [2]). The interaction between the viscous and inviscid effects is considered, and some of the effects of low density are accounted for by using boundary conditions which allow a slip velocity and a temperature jump at the wall. Expressions are obtained for the transient shear stress and heat transfer at the wall.

1.2 Review of Pertinent Literature

The presence of a slip velocity at the interface between a solid and a gas at low pressures was first deduced by Kundt and Warburg [3] at the turn of the century, when they observed that the damping of an oscillating disk by the surrounding gas decreased with the pressure. Soon afterward Maxwell [4] developed an approximate analysis of a monatomic gas adjacent to an isothermal surface, showing a relationship between the slip velocity and the gradient of the tangential flow velocity,

$$u_w = A\lambda_w \left(\frac{\partial u}{\partial y}\right)_w . \tag{1.2}$$

More refined calculations, involving approximate solutions to the Maxwell-Boltzmann equation in the vicinity of the wall, yield the results given by Kennard [5],

$$u_w = A\lambda_w \left(\frac{\partial u}{\partial y}\right)_w + \frac{3}{4}\frac{\nu_w}{\theta_w}\left(\frac{\partial \theta}{\partial x}\right)_w \tag{1.3}$$

and

$$\theta_w = \theta_p + B\lambda_w \left(\frac{\partial \theta}{\partial y}\right)_w - \frac{1}{2}\frac{\nu_w}{R\theta_w} u_w \left(\frac{\partial \theta}{\partial x}\right)_w . \tag{1.4}$$

The second term in equation (1.3) indicates that a temperature gradient along a surface induces a flow in the direction of increasing temperature. This phenomenon is known as "thermal creep" [6] and is usually of little significance when velocity gradients at the wall are large. It will be neglected in the present work.

Equation (1.4) is known as the "temperature jump" boundary condition - a condition which exists in rarefied slip flow [5]. For the normal-density continuum regime the gas temperature at the wall is equal to the wall temperature. However, if the gas is slightly rarefied, the gas temperature at the surface can differ from the wall temperature. A theory describing this temperature difference was first derived by Poisson [7].

A question which is of prime importance when considering low-density gases is the applicability of the continuum Navier-Stokes equations. Kogan [8] showed that Prandtl's boundary layer formulation

is not only consistent with boundary conditions (1.3) and (1.4), but also provides slip solutions at least as accurate as any produced by the higher approximations such as the Burnett or Thirteen Moment equations. Kogan noted that "The Burnett equations permit us to obtain a more accurate description of the flow in a region where the Navier-Stokes equations are also applicable, but they do not permit us to progress into the region where the latter are inapplicable". This finding agrees with the conclusions of Schaaf and Chambré [9] wherein they state that the Navier-Stokes formulation is the best available. A solution of the Maxwell-Boltzmann equation, while being the most realistic, is usually too complex to be practical.

Several special problems in slip flow, such as the Couette flow between flat plates and concentric cylinders, stagnation point flow, and the case of an incompressible boundary layer [10] have been examined during the last two decades. In Ref. [11], slip flow over a flat plate is discussed from another point of view by analogy with the first Stokes problem of an infinite flat plate suddenly set into steady motion in a viscous fluid. This approach is relatively simple, but is limited by questions of the validity of the analogy for slip flow.

Maslen [12] obtained the solution to the problem of slip flow in a compressible boundary layer on a flat plate by introducing a perturbation of the known continuum solution (the Blasius problem). The boundary layer equations are expanded in powers of the small parameter ε, where

$$\varepsilon = \frac{Me \sqrt{\gamma}}{\sqrt{Re_x}} , \tag{1.5}$$

to give two sets of equations, one of zero-order in ε, and the other containing the first order quantities. The same procedure is followed with the slip boundary conditions (1.3) and (1.4). The zero-order system is seen to be the usual continuum boundary-layer system to which the solution is known, and the first-order system is solved explicitly in terms of the zero-order quantities. This technique was later applied by Aroesty [13] in order to introduce slip effects to the strong-interaction problem of Li and Nagamatsu [14]. Oguchi [15] combined the linear perturbation and Karman-Pohlhausen methods to obtain a simpler and more useful solution to Maslen's problem.

Recently Libby and Chen [16] presented a unique solution to the laminar boundary layer equation with uniform injection of mass at the plate surface; the flow in this case is non-similar. Wazzan, Lind, and Liu [17] have found it possible to apply the same technique to the problem of slip at the wall. The solution is obtained by a series expansion in terms of a slip parameter $\sigma(S)$, where

$$f'(S,\eta=0) = \frac{u_w(S)}{U_e} \equiv \sigma(S) \;\; ; \;\; S \equiv S(x) \; . \tag{1.6}$$

With the above approach $\sigma(S)$ becomes an independent variable so that the results can be applied to any value of $\sigma(S)$ desired. Since the method requires that $\sigma(S)$ be known a priori, it is not suitable for use with the classical slip boundary conditions (1.3) and (1.4).

The method chosen for the present work is a modification of the technique used by Gupta and Rodkiewicz [18] for unsteady non-slip flow with strong interactions. Because the flow-field with slip is non-similar, it is found necessary to introduce an x-dependant variable ξ into the semi-similar (η,τ) coordinate system used by Gupta. The differential equations thus obtained are subjected to the classical slip boundary conditions and are solved by an iterative numerical scheme due to Smith and Clutter [19].

CHAPTER II

EQUATIONS GOVERNING THE FLOW

2.1 Boundary Layer Equations

We consider the unsteady, two-dimensional flow of a slightly rarefied compressible viscous fluid over a semi-infinite flat plate. The coordinate system is attached to the plate, with the origin at the leading edge. Using the usual boundary layer approximations, the conservation equations reduce [19] to the following:

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \; ; \tag{2.1}$$

Conservation of Momentum

x:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \; ; \tag{2.2}$$

y:

$$\frac{\partial P}{\partial y} = 0 \; ; \tag{2.3}$$

Conservation of Energy

$$\rho\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = \frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 . \tag{2.4}$$

These equations may be specialized for the region outside the boundary layer, where viscous and heat conduction effects are negligible. In this region the velocity, temperature, pressure and density depend only on x and t. Then

$$\frac{d}{dx}(\rho_e U_e) = 0 , \tag{2.1a}$$

$$\rho_e U_e \frac{dU_e}{dx} = - \frac{dP_e}{dx} , \tag{2.2a}$$

and

$$\rho_e \frac{dh_e}{dx} = \frac{dP_e}{dx} \tag{2.4a}$$

where the hypersonic assumption (1.1) has been used to remove the time-derivatives of ρ_e, U_e, h_e, and P_e.

Thus the boundary conditions on equations (2.1) through (2.4) are

$$\begin{aligned} y = 0: \qquad & u(x,0,t) = u_w(x,t) , \\ & v(x,0,t) = 0 , \\ & h(x,0,t) = h_w(x,t) , \end{aligned} \tag{2.5}$$

and for an insulated wall,

$$\left.\frac{\partial h}{\partial y}\right|_{y=0} = - Pr \, u_w \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$y = y_e: \qquad u(x,y_e,t) = U_e(x) \ ,$$

$$h(x,y_e,t) = h_e(x) \ , \tag{2.6}$$

and

2.2 Transforming the Boundary Layer Equations

The previous section has shown our problem to be one which requires the solution of coupled partial differential equations. There are five dependent variables u, v, p, ρ, and h, which are functions of the independent variables x, y, and t. The complexity of this problem makes a direct solution of equations (2.1) through (2.4) a difficult task. It is possible, however, to simplify the problem by using a series of coordinate transformations. The end result is a pair of coupled partial differential equations which are more amenable to a numerical solution because the number of dependent variables has been reduced from five to two. Since no similarity transformation was found, we still have three independent variables.

2.2.a Restricted Dorodnitsyn-Howarth Transformation

The explicit dependence of the boundary-layer equations on density ρ may be partially eliminated by introducing the Dorodnitsyn-Howarth transformation as applied to the y-variable. Thus, when the new independent variables

$$\bar{x} = x \ ; \quad \bar{y} = \int_0^y \frac{\rho}{\rho_{ref}}\, dy \ ; \quad \bar{t} = t \tag{2.7}$$

are used, the partial derivatives with respect to the old variables

become

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial x} \frac{\partial}{\partial \bar{y}} ,$$

$$\frac{\partial}{\partial y} = \frac{\rho}{\rho_{ref}} \frac{\partial}{\partial \bar{y}} , \tag{2.8}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \frac{\partial \bar{y}}{\partial t} \frac{\partial}{\partial \bar{y}} .$$

When these relations are substituted into equations (2.2) and (2.4), we obtain, respectively,

$$\rho[\frac{\partial u}{\partial \bar{t}} + u \frac{\partial u}{\partial \bar{x}} + (\frac{\partial \bar{y}}{\partial t} + u \frac{\partial \bar{y}}{\partial x} + \frac{\rho}{\rho_{ref}} v) \frac{\partial u}{\partial \bar{y}}]$$

$$= - \frac{\partial P}{\partial \bar{x}} + \frac{\rho}{\rho_{ref}} \frac{\partial}{\partial \bar{y}} (\frac{\mu\rho}{\rho_{ref}} \frac{\partial u}{\partial \bar{y}}) , \tag{2.9}$$

and

$$\rho[(\frac{\partial H}{\partial \bar{t}} + \frac{\partial H}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial t}) + u(\frac{\partial H}{\partial \bar{x}} + \frac{\partial H}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial x}) + \frac{\rho}{\rho_{ref}} v \frac{\partial H}{\partial \bar{y}}]$$

$$= \frac{\rho}{\rho_{ref}} \frac{\partial}{\partial \bar{y}} [\frac{\mu\rho}{\rho_{ref} Pr} \frac{\partial H}{\partial \bar{y}} + \frac{\mu\rho}{\rho_{ref} Pr} (Pr-1) u \frac{\partial u}{\partial \bar{y}}] . \tag{2.10}$$

Equations (1.1) and (2.3) together imply that $\frac{\partial P}{\partial t} = 0$.

The derivation of equation (2.10) requires the use of equations (2.2) and (2.3) in (2.4). The relation

$$h = H - \frac{u^2}{2} ,$$

which is a consequence of the hypersonic assumption, is also used in the derivation.

At this point a stream function ψ is introduced such that

$$u = \frac{\partial\psi}{\partial y} \quad . \tag{2.12}$$

From equation (2.7) we have

$$\rho = \rho_{ref} \frac{\partial\bar{y}}{\partial y} \quad . \tag{2.13}$$

From (2.8) and (2.12),

$$\rho u = \rho_{ref} \frac{\partial\psi}{\partial y} \quad . \tag{2.14}$$

This relation plus equations (2.8) and (2.13) are substituted into the continuity equation (2.1). Then (2.1) may be integrated to yield

$$v = - \frac{\rho_{ref}}{\rho} \left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial\bar{y}} \frac{\partial\bar{y}}{\partial x} + \frac{\partial\bar{y}}{\partial t}\right) \tag{2.15}$$

if the assumption is made that both $\bar{y}$ and ψ are continuous functions which are continuously differentiable at least to second order [20]. Boundary condition (2.5) is used to show that the constant of integration is zero.

If ψ is defined by equations (2.12) and (2.15), then the

continuity equation (2.1) is satisfied. Use of the stream function causes the momentum and energy equations to become, respectively,

$$\frac{\partial^2\psi}{\partial\bar{y}\partial\bar{t}} + \frac{\partial\psi}{\partial\bar{y}}\frac{\partial^2\psi}{\partial\bar{y}\partial\bar{x}} - \frac{\partial\psi}{\partial\bar{x}}\frac{\partial^2\psi}{\partial\bar{y}^2}$$

$$= -\frac{1}{\rho}\frac{\partial P}{\partial\bar{x}} + \frac{1}{\rho_{ref}}\frac{\partial}{\partial\bar{y}}\left(\frac{\mu\rho}{\rho_{ref}}\frac{\partial^2\psi}{\partial\bar{y}^2}\right) \tag{2.16}$$

and

$$\frac{\partial H}{\partial\bar{t}} + \frac{\partial\psi}{\partial\bar{y}}\frac{\partial H}{\partial\bar{x}} - \frac{\partial H}{\partial\bar{y}}\frac{\partial\psi}{\partial\bar{x}}$$

$$= \frac{1}{\rho_{ref}}\frac{\partial}{\partial\bar{y}}\left[\frac{\mu\rho}{\rho_{ref}Pr}\frac{\partial H}{\partial\bar{y}} + \frac{\mu\rho}{\rho_{ref}Pr}(Pr-1)\frac{\partial\psi}{\partial\bar{y}}\frac{\partial^2\psi}{\partial\bar{y}^2}\right] . \tag{2.17}$$

2.2b Howarth-Stewartson Transformation

The explicit appearance of ρ in equations (2.16) and (2.17) may be removed by further transformations. The new independent variables are

$$\bar{X} = \int_0^{\bar{x}} C\left(\frac{a_e}{a_{ref}}\right)\left(\frac{P_e}{P_{ref}}\right) d\bar{x} \ , \ \bar{Y} = \frac{a_e}{a_{ref}}\bar{y} \ , \ \bar{T} = \bar{t} \ . \tag{2.18}$$

Derivatives can be transformed via

$$\frac{\partial}{\partial\bar{x}} = C\left(\frac{a_e}{a_{ref}}\right)\left(\frac{P_e}{P_{ref}}\right)\frac{\partial}{\partial\bar{X}} + \frac{\partial\bar{Y}}{\partial\bar{x}}\frac{\partial}{\partial\bar{Y}} ,$$

$$\frac{\partial}{\partial\bar{y}} = \frac{a_e}{a_{ref}}\frac{\partial}{\partial\bar{Y}} , \tag{2.19}$$

$$\frac{\partial}{\partial \overline{t}} = \frac{\partial}{\partial \overline{T}} ,$$

where C, which has the form of the Chapman-Rubesin constant [21], is in this case not a constant, but is a function of x appearing in the linear viscosity law,

$$C \equiv C(x) = \frac{\mu \theta_{ref}}{\mu_{ref} \theta} . \tag{2.20}$$

These relations, when used in equations (2.16) and (2.17), yield

$$\frac{a_e}{a_{ref}} \frac{\partial^2 \psi}{\partial \overline{Y} \partial \overline{T}} + C\left(\frac{a_e}{a_{ref}}\right)^3 \left(\frac{P_e}{P_{ref}}\right) \frac{\partial \psi}{\partial \overline{Y}} \frac{\partial^2 \psi}{\partial \overline{X} \partial \overline{Y}} + C\left(\frac{a_e}{a_{ref}}\right)^3 \left(\frac{P_e}{P_{ref}}\right) \frac{1}{a_e} \frac{\partial a_e}{\partial \overline{X}} \left(\frac{\partial \psi}{\partial \overline{Y}}\right)^2$$
$$- C\left(\frac{a_e}{a_{ref}}\right)^3 \left(\frac{P_e}{P_{ref}}\right) \frac{\partial \psi}{\partial \overline{X}} \frac{\partial^2 \psi}{\partial \overline{Y}^2} = - \frac{C}{\rho} \left(\frac{a_e}{a_{ref}}\right)\left(\frac{P_e}{P_{ref}}\right) \frac{\partial P}{\partial \overline{X}}$$
$$+ C\left(\frac{a_e}{a_{ref}}\right)\left(\frac{P_e}{P_{ref}}\right) \frac{\mu_{ref}}{\rho_{ref}} \frac{\partial^3 \psi}{\partial \overline{Y}^3} \tag{2.21}$$

and

$$\frac{\partial H}{\partial \overline{T}} + C\left(\frac{a_e}{a_{ref}}\right)^2 \left(\frac{P_e}{P_{ref}}\right) \frac{\partial \psi}{\partial \overline{Y}} \frac{\partial H}{\partial \overline{X}} - C\left(\frac{a_e}{a_{ref}}\right)^2 \left(\frac{P_e}{P_{ref}}\right) \frac{\partial \psi}{\partial \overline{X}} \frac{\partial H}{\partial \overline{Y}}$$
$$= \frac{1}{\rho_{ref}} \left(\frac{a_e}{a_{ref}}\right)^2 \frac{\partial}{\partial \overline{Y}} \left[\frac{\mu \rho}{\rho_{ref} Pr} \frac{\partial H}{\partial \overline{Y}} + \frac{\mu \rho}{\rho_{ref} Pr} (Pr-1)\left(\frac{a_e}{a_{ref}}\right)^2 \frac{\partial \psi}{\partial \overline{Y}} \frac{\partial^2 \psi}{\partial \overline{Y}^2}\right] . \tag{2.22}$$

The equation of state for a perfect gas,

$$P = \rho R \theta \ , \tag{2.23}$$

was required in the derivation of (2.21).

2.2c Temporal Transformation

Gupta and Rodkiewicz [18] found that the conservation equations (2.21) and (2.22) could be simplified by a distortion of the time variable. If the transformation

$$X = \overline{X} \ , \quad Y = \overline{Y} \ , \quad T = \int_0^{\overline{T}} C\left(\frac{a_e}{a_{ref}}\right)^2 \left(\frac{P_e}{P_{ref}}\right) d\overline{T} \ , \tag{2.24}$$

is introduced, then

$$\frac{\partial}{\partial \overline{X}} = \frac{\partial}{\partial X} + \frac{\partial T}{\partial \overline{X}} \frac{\partial}{\partial T} \ ,$$

$$\frac{\partial}{\partial \overline{Y}} = \frac{\partial}{\partial Y} \ , \tag{2.25}$$

$$\frac{\partial}{\partial \overline{T}} = C\left(\frac{a_e}{a_{ref}}\right)^2 \left(\frac{P_e}{P_{ref}}\right) \frac{\partial}{\partial T} \ .$$

Thus equations (2.21) and (2.22) become, respectively,

$$\frac{\partial^2 \psi}{\partial Y \partial T} + \frac{\partial \psi}{\partial Y} \frac{\partial^2 \psi}{\partial Y \partial X} - \frac{\partial \psi}{\partial X} \frac{\partial^2 \psi}{\partial Y^2} - \frac{H}{H_e} a_{ref}^2 M_e \frac{dM_e}{dX}$$

$$+ \frac{\partial T}{\partial \overline{X}} \left(\frac{\partial \psi}{\partial Y} \frac{\partial^2 \psi}{\partial Y \partial T} - \frac{\partial \psi}{\partial T} \frac{\partial^2 \psi}{\partial Y^2}\right) = \nu_{ref} \frac{\partial^3 \psi}{\partial Y^3} \tag{2.26}$$

and

$$\frac{\partial H}{\partial T} + \frac{\partial \psi}{\partial Y}\frac{\partial H}{\partial X} - \frac{\partial \psi}{\partial X}\frac{\partial H}{\partial Y} + \frac{\partial T}{\partial \overline{X}}\left(\frac{\partial \psi}{\partial Y}\frac{\partial H}{\partial T} - \frac{\partial \psi}{\partial T}\frac{\partial H}{\partial Y}\right)$$

$$= \frac{\nu_{ref}}{Pr}\frac{\partial}{\partial Y}\left[\frac{\partial H}{\partial Y} + (Pr-1)\left(\frac{a_e}{a_{ref}}\right)^2 \frac{\partial \psi}{\partial Y}\frac{\partial^2 \psi}{\partial Y^2}\right] , \tag{2.27}$$

where equation (2.11) plus the following relations have been used:

$$a_e^{\,2} = (\gamma-1)\, h_e , \tag{2.28}$$

$$\frac{1}{a_e}\frac{\partial a_e}{\partial X}\left(\frac{\partial \psi}{\partial Y}\right)^2 = \left(\frac{a_{ref}}{a_e}\right)^2 \frac{1}{h_e}\frac{\partial h_e}{\partial X}\frac{u^2}{2} , \tag{2.29}$$

$$\frac{1}{\rho}\left(\frac{a_{ref}}{a_e}\right)^2 \frac{\partial P}{\partial X} = \left(\frac{a_{ref}}{a_e}\right)^2 \frac{h}{h_e}\frac{\partial h_e}{\partial X} , \tag{2.30}$$

$$\frac{U_e^{\,2}}{h_e} = (\gamma-1)\, M_e^{\,2} , \tag{2.31}$$

$$\left(\frac{a_{ref}}{a_e}\right)^2 \frac{1}{h_e}\frac{\partial h_e}{\partial X} = - \frac{a_{ref}^2}{H_e} M_e \frac{dM_e}{dX} . \tag{2.32}$$

Equations (2.26) and (2.27) were obtained by Gupta and Rodkiewicz [18] and are the flow governing equations in terms of the independent variables X, Y, and T. These new variables are related to the physical variables x, y, and t by the following expressions:

$$X = \int_0^x C\left(\frac{a_e}{a_{ref}}\right)\left(\frac{P_e}{P_{ref}}\right) dx \ , \ Y = \frac{a_e}{a_{ref}} \int_0^y \frac{\rho}{\rho_{ref}} \ dy \ ,$$

$$T = C\left(\frac{a_e}{a_{ref}}\right)^2 \left(\frac{P_e}{P_{ref}}\right) t \quad . \tag{2.33}$$

2.2d Final Transformation

Gupta was able to further simplify equations (2.26) and (2.27) by applying a similarity transformation. In the present problem, because of the slip boundary conditions at the wall, the velocity and temperature of the gas adjacent to the wall do not remain constant, but instead are functions of x and t. In this case a similarity transformation of the form found by Rodkiewicz and Reshotko [1] and used by Gupta [22] leads to boundary conditions which are functionally inconsistent; the x- and t-dependence can not be eliminated. In Appendix A the boundary conditions are examined, with the result that the following coordinate transformations are suggested:

$$\xi = \frac{1}{\lambda_{ref}L}\left[\frac{2\nu_{ref}X}{(m+1)V_e}\right]^{1/2} \ , \ \eta = \left[\frac{(m+1)V_e}{2\nu_{ref}X}\right]^{1/2} Y \ , \ \tau = \frac{V_e T}{X} \ , \tag{2.34}$$

where

$$L \equiv L(X,T) = \frac{\lambda_w \rho_w a_e}{\lambda_{ref}\rho_{ref}a_{ref}} \tag{2.35}$$

and

$$V_e \equiv V_e(X) = \frac{a_{ref}}{a_e} U_e = a_{ref} M_e \ . \tag{2.36}$$

We let

$$S \equiv S(\xi,\eta,\tau) = \frac{H}{H_e} \quad . \tag{2.37}$$

The derivatives appearing in (2.26) and (2.27) are then transformed as follows:

$$\frac{\partial}{\partial X} = \frac{\partial\xi}{\partial X}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial X}\frac{\partial}{\partial\eta} + \frac{\partial\tau}{\partial X}\frac{\partial}{\partial\tau} \quad ,$$

$$\frac{\partial}{\partial Y} = \frac{\partial\eta}{\partial Y}\frac{\partial}{\partial\eta} \, , \tag{2.38}$$

$$\frac{\partial}{\partial T} = \frac{\partial\xi}{\partial T}\frac{\partial}{\partial\xi} + \frac{\partial\tau}{\partial T}\frac{\partial}{\partial\tau} \, .$$

We now define a new stream function f such that

$$\psi = \left[\frac{2\nu_{ref}XV_e}{(m+1)}\right]^{1/2} f(\xi,\eta,\tau) \; . \tag{2.39}$$

When equations (2.34) and (2.39) are used in (2.38), the following expressions result:

$$\frac{\partial\psi}{\partial Y} = V_e \; f' \tag{2.40a}$$

$$\frac{\partial^2\psi}{\partial Y\partial T} = \frac{V_e^{\,2}}{X} \; \dot{f}' \; , \tag{2.40b}$$

$$\frac{\partial\psi}{\partial X} = \frac{1}{2} Yf' \frac{dV_e}{dX} - \frac{1}{2} Yf' \frac{V_e}{X}$$

$$+ \sqrt{\frac{2\nu_{ref}}{(m+1)}} X^{1/2} V_e^{1/2} \dot{f}(- \frac{\tau}{X} + \frac{\tau}{V_e} \frac{dV_e}{dX}$$

$$+ \frac{1}{2} \sqrt{\frac{2\nu_{ref}}{(m+1)}} \frac{V_e^{1/2}}{X^{1/2}} f + \frac{1}{2} \sqrt{\frac{2\nu_{ref}}{(m+1)}} \frac{X^{1/2}}{V_e^{1/2}} f \frac{dV_e}{dX}$$

$$+ \frac{m+1}{2} \sqrt{\frac{2\nu_{ref}}{(m+1)}} \frac{V_e^{1/2}}{X^{1/2}} \xi f^* , \qquad (2.40c)$$

$$\frac{\partial^2\psi}{\partial X\partial Y} = f' \frac{dV_e}{dX} + \frac{1}{2} \sqrt{\frac{(m+1)}{2\nu_{ref}}} \frac{V_e^{1/2}}{X^{1/2}} Yf'' \frac{dV_e}{dX}$$

$$- \frac{1}{2} \sqrt{\frac{(m+1)}{2\nu_{ref}}} \frac{V_e^{3/2}}{X^{3/2}} Yf'' + V_e \dot{f}'(- \frac{\tau}{X} + \frac{\tau}{V_e} \frac{dV_e}{dX})$$

$$+ \frac{m+1}{2} \frac{V_e}{X} \xi f^{*\prime} , \qquad (2.40d)$$

$$\frac{\partial^2\psi}{\partial Y^2} = \sqrt{\frac{(m+1)}{2\nu_{ref}}} \frac{V_e^{3/2}}{X^{1/2}} f'' , \qquad (2.40e)$$

$$\frac{\partial^3\psi}{\partial Y^3} = \frac{(m+1)}{2\nu_{ref}} \frac{V_e^2}{X} f''' , \qquad (2.40f)$$

$$\frac{\partial\psi}{\partial T} = \sqrt{\frac{2\nu_{ref}}{(m+1)}} \frac{V_e^{3/2}}{X^{1/2}} \dot{f} , \qquad (2.40g)$$

$$\frac{\partial T}{\partial X} = - T(\frac{2\gamma-1}{\gamma-1}) \frac{a_e^2}{H_e} M_e \frac{dM_e}{dX} , \qquad (2.40h)$$

$$\frac{\partial H}{\partial T} = H_e \dot{S} \frac{V_e}{X} , \qquad (2.40i)$$

$$\frac{\partial H}{\partial X} = \sqrt{\frac{(m+1)}{2\nu_{ref}}}\, H_e S'\left(\frac{1}{2}\frac{Y}{\sqrt{XV_e}}\frac{dV_e}{dX} - \frac{1}{2}\, Y\, \frac{V_e^{\;1/2}}{X^{3/2}}\right)$$

$$+ H_e\, \dot{S}\left(-\frac{\tau}{X} + \frac{\tau}{V_e}\frac{\partial V_e}{\partial X}\right) + S\,\frac{\partial H_e}{\partial X}$$

$$+ H_e\, \frac{m+1}{2}\, \frac{\xi}{X}\, S^* \,, \tag{2.40j}$$

$$\frac{\partial H}{\partial Y} = \sqrt{\frac{(m+1)}{2\nu_{ref}}}\, \frac{V_e^{\;1/2}}{X^{1/2}}\, H_e\, S' \,, \tag{2.40k}$$

$$\frac{\partial^2 H}{\partial Y^2} = \frac{m+1}{2\nu_{ref}}\, H_e\, \frac{V_e}{X}\, S'' \,. \tag{2.40l}$$

If equations (2.34) through (2.40) are substituted into the momentum and energy equations, then the result is

$$\frac{2\nu_{ref}}{m+1}\,\dot{f}' + \frac{2\nu_{ref}}{m+1}\,\frac{X}{V_e}\frac{dV_e}{dX}\, f^{2\prime} - \frac{2\nu_{ref}}{m+1}\,\frac{X}{V_e}\frac{dV_e}{dX}\frac{H}{H_e}$$

$$- \frac{2\nu_{ref}}{m+1}\left(\frac{1}{2} + \frac{1}{2}\frac{X}{V_e}\frac{dV_e}{dX}\right) ff'' - \frac{2\nu_{ref}}{m+1}\, V_e T\Big[\frac{1}{X} - \frac{1}{V_e}\frac{dV_e}{dX}$$

$$+ \left(\frac{2\gamma-1}{\gamma-1}\right)\frac{a_e^{\;2}}{H_e}\, M_e\, \frac{dM_e}{dX}\Big](f'\dot{f}' - f\dot{f}'') - \nu_{ref} f'''$$

$$= \frac{2\nu_{ref}}{m+1}\,\frac{m+1}{2}\,\xi(f^*f'' - f'f^{*\prime}) \tag{2.41}$$

and

$$\frac{2\nu_{ref}}{m+1}\dot{S} + \frac{2\nu_{ref}}{m+1}\ \tau\ [1 - \frac{X}{V_e}\frac{dV_e}{dX} + (\frac{2\gamma-1}{\gamma-1})\ \frac{a_e^{\ 2}}{H_e} M_e \frac{dM_e}{dX}\ X](S'\dot{f}-\dot{S}f')$$

$$- \frac{2\nu_{ref}}{m+1}(\frac{1}{2} + \frac{1}{2}\frac{X}{V_e}\frac{dV_e}{dX})S'f - \frac{\nu_{ref}}{a_{ref}^{\ 2}}\frac{Pr-1}{Pr}\frac{a_e^{\ 2}}{H_e} V_e^{\ 2}(f^{2}{}''+f'f''') - \frac{\nu_{ref}}{Pr}S''$$

$$= \frac{2\nu_{ref}}{m+1}\frac{m+1}{2}\ \xi(f^*S'-f'S^*)\ . \tag{2.42}$$

Now let

$$V_e = eX^m \tag{2.43}$$

and

$$\beta = \frac{2m}{m+1}\ . \tag{2.44}$$

Then, since H_e is independent of X, and since

$$\frac{U_e^{\ 2}}{H_e} = \frac{(\gamma-1)\ M_e^{\ 2}}{1 + \frac{\gamma-1}{2} M_e^{\ 2}}\ , \tag{2.45}$$

equations (2.41) and (2.42) may be written as

$$(2-\beta)\dot{f}' + \beta(f'^{2}-S) - 2(1-\beta)\tau(f'\dot{f}'-\dot{f}f'')$$

$$- \beta(\frac{2\gamma-1}{\gamma-1})\ \tau\{\frac{(\gamma-1)\ M_e^{\ 2}}{1 + \frac{\gamma-1}{2} M_e^{\ 2}}\}\ (f'\dot{f}'-\dot{f}f'') - ff'' - f'''$$

$$= \xi(f^*f''-f'f^{*\prime}) \tag{2.46}$$

and

$$\dot{S}\,Pr[(2-\beta)-2(1-\beta)\tau f'] - S'' - S'Pr[f-2(1-\beta)\tau\dot{f}]$$

$$- \beta\left(\frac{2\gamma-1}{\gamma-1}\right) Pr\tau \left\{\frac{(\gamma-1)\,M_e^{\,2}}{1+\frac{\gamma-1}{2}M_e^{\,2}}\right\} (f'\dot{S}-\dot{f}S')-(Pr-1)\left\{\frac{(\gamma-1)\,M_e^{\,2}}{1+\frac{\gamma-1}{2}M_e^{\,2}}\right\}$$

$$\times\,(f''^2+f'f''') = Pr\,\xi(f^*S'-f'S^*)\ . \qquad (2.47)$$

Since the use of the hypersonic assumption limits us to the consideration of large Mach number flows only, we may safely use the approximation

$$\left\{\frac{(\gamma-1)\,M_e^{\,2}}{1+\frac{\gamma-1}{2}M_e^{\,2}}\right\} \approx 2\ . \qquad (2.48)$$

This allows us to write equations (2.46) and (2.47) as

$$(2-\beta)\dot{f}' + \beta(f'^2-S) - 2\tau\left[\left(\frac{\gamma}{\gamma-1}\right)\beta+1\right](f'\dot{f}'-\dot{f}f'')$$

$$- ff'' - f''' = \xi(f^*f''-f'f^{*\prime}) \qquad (2.49)$$

and

$$(2-\beta)\,Pr\,\dot{S} - 2Pr\,\tau\left[\left(\frac{\gamma}{\gamma-1}\right)\beta+1\right](f'\dot{S}-\dot{f}S') - PrfS'$$

$$- S'' - 2(Pr-1)(f''^2+f'f''') = Pr\,\xi(f^*S'-f'S^*)\ . \qquad (2.50)$$

These two expressions are the final form of the governing equations

for the unsteady, two-dimensional high-speed flow of a slightly rarefied compressible viscous fluid over a semi-infinite flat plate.

The energy equation, when specialized for Pr = 1, becomes

$$(2-\beta)\dot{S} - 2\tau[(\frac{\gamma}{\gamma-1})\beta+1](f'\dot{S}-\dot{f}S') - fS' - S''$$

$$= \xi(f^*S'-f'S^*) \ . \qquad (2.51)$$

2.3 Specialization for Zero Pressure Gradient

Equations (2.49) and (2.51) are valid for any pressure gradient, but since the value of β is assumed to be independent of x, the analysis must be limited to flows in which the pressure gradient parameter β is a constant. There are only two types of flow which meet this requirement - the strong interaction case, for which we must have $\beta = (\gamma-1)/\gamma$ [23], and the weak interaction case, with $\beta = 0$. It is known that close to the leading edge of the plate, the interaction between the viscous and inviscid flow regions has a strong effect on the flow properties. The interaction effects become weaker as the distance from the leading edge increases. We thus see that the two values of β given above represent only the extremes of viscous interaction. There must exist a region on the plate for which the proper value of β lies between 0 and $(\gamma-1)/\gamma$. Since the formulation of the problem does not allow us to give β any sort of dependence on x, we are forced to deal with one of the two extremes. The weak interaction case (β=0) is used in the remainder of this work. There are two reasons for this choice:

1. Strong interactions are important in a region of only vaguely defined size. Although it is possible for the flow properties and wing geometry to be such that strong interactions exist over the entire wing surface, it is more likely that a wing producing useful lift would have a chord long enough to allow an appreciable weakening of the viscous interactions. At speeds in the low hypersonic range (Mach 5 to Mach 15) the region of strong interactions should be but a small percentage of the total wing length. The observations of the authors of Ref. [24] indicate that the strong interaction region should be important only for flight Mach numbers greater than about 15.
2. Solving the coupled strong interaction equations requires computations which are almost seven times as lengthy as for the weak interaction case.

Equations (2.49) and (2.51), when specialized for zero pressure gradient, have the form

$$2\dot{f}' - 2\tau(f'\dot{f}'-\dot{f}f'') - ff'' - f''' = \xi(f^*f''-f'f^{*\prime}) \qquad (2.52)$$

and

$$2\dot{S} - 2\tau(f'\dot{S}-\dot{f}S') - fS' - S'' = \xi(f^*S'-f'S^*) \; . \qquad (2.53)$$

The solution of these two equations is described in the following chapters.

We should take note of the fact that these equations have been developed using an arbitrary reference condition. The only assumption which has been made is that the reference values are independent of x. If no longitudinal pressure gradient exists (β=0) the free stream Mach number is independent of x, and thus the free stream properties are also constant and may be used as the reference values. In the presence of a longitudinal pressure gradient, the free stream Mach number is a function of x, as are the free stream properties. However, the free stream stagnation properties remain constant, and hence are ideal for use as the reference values.

2.4 Boundary and Initial Conditions

The boundary conditions appropriate to equations (2.52) and (2.53) are as follows:

1. When $y = y_e$ we have $Y = Y_e$, $\eta = \eta_e$, and $u = U_{e3}$. Then

$$f'(\xi,\eta_e,\tau) = \frac{1}{V_{e3}} \frac{a_{ref}}{a_{e3}} u = \frac{u}{U_{e3}} = 1 \ . \tag{2.54}$$

Also, we have $H = H_e$, from which

$$S(\xi,\eta_e,\tau) = 1 \ . \tag{2.55}$$

2. When $y = 0$, we have $Y = 0$, $\eta = 0$, and $u = A\lambda_w \frac{\partial u}{\partial y}$. Thus, from Appendix A,

$$f'(\xi,0,\tau) = \frac{A}{\xi} f''(\xi,0,\tau) \ , \tag{2.56}$$

so long as $\xi > \delta\xi$ and $\tau \neq 0$. Here $\delta\xi$ is that value of ξ for which $f'(\delta\xi,0,\tau) = \frac{A}{\delta\xi} f''(\delta\xi,0,\tau) = 1$.

These restrictions are necessary because of the unrealistic behavior of the boundary condition at $\xi = 0$ and the inconsistency with condition (2.60) at $\tau = 0$.

When $y = 0$, we have $Y = 0$, $\eta = 0$, and $\psi = 0$. Hence

$$f(\xi,0,\tau) = 0. \tag{2.57}$$

At the plate surface (y=0) we have $\theta = \theta_P + B\lambda_w \left.\frac{\partial\theta}{\partial y}\right|_w$.

Then, from Appendix A,

$$S(\xi,0,\tau) = \frac{\theta_P}{\theta_0} + \frac{B}{\xi} S'(\xi,0,\tau) + \frac{(A^2-2AB)}{\xi^2} f''^2(\xi,0,\tau) \tag{2.58}$$

so long as $\xi > \delta\xi$ and $\tau \neq 0$.

For the case of no wall heat transfer, Appendix A shows that

$$S'(\xi,0,\tau)_{ad} = 0. \tag{2.59}$$

3. When $t = 0$, we have $\tau = 0$, $u = u_1 + (U_{e3}-U_{e1})$*, and hence

$$f'(\xi,\eta,0) = f_1'(\xi_1,\eta_1) \frac{U_{e1}}{U_{e3}} + (1 - \frac{U_{e1}}{U_{e3}}) . \tag{2.60}$$

---

*$u_1 = u(\xi,\eta,\tau<0)$; $u_2 = u(\xi,\eta,\tau=0)$; $u_3 = u(\xi,\eta,\tau>0)$

Also at $t = 0$, we have $H = H_2$. Then

$$S(\xi,\eta,0) = \frac{H_2}{H_{e3}} . \qquad (2.61)$$

4. The assumption is made that free-stream conditions exist at the leading edge of the plate. Thus, at $\xi = \delta\xi$,

$$f'(\delta\xi,\eta,\tau) = 1, \; S(\delta\xi,\eta,\tau) = 1 . \qquad (2.62)$$

2.5 Correlation with Equations in the Literature

When equations (2.52) and (2.53) are specialized for steady state, the result is

$$f''' + ff'' + \xi(f^*f''-f'f^{*\prime}) = 0 \qquad (2.63)$$

and

$$S'' + fS' + \xi(f^*S'-f'S^*) = 0 . \qquad (2.64)$$

Equation (2.63) has been obtained by Wazzan, Lind, and Liu [17] in a slightly different form. A coupled form of equations (2.63) and (2.64) was obtained and solved by G. Berard [25]. A comparison with his experimental results is included in Chapter V.

For the case where there is no slip or temperature jump at the wall, f and S do not depend on ξ. Thus the derivatives denoted by

"*" are non-existent, and equations (2.52) and (2.53) become

$$2\dot{f}' - 2\tau(f'\dot{f}'-\dot{f}f'') - ff'' - f''' = 0 \tag{2.65}$$

and

$$2\dot{S} - 2\tau(f'\dot{S}-\dot{f}S') - fS' - S'' = 0 \;. \tag{2.66}$$

If we now define a new function

$$g(\eta,\tau) = (1 + \frac{\gamma-1}{2} M_e^2)\, S(\eta,\tau) - \frac{\gamma-1}{2} M_e^2 f'^2(\eta,\tau) \tag{2.67}$$

and substitute it into (2.66), we find that

$$\begin{aligned} &2\dot{g}(1-\tau f') - g'' - g'(f-2\tau\dot{f}) - (\gamma-1)\, M_e^2 f''^2 \\ &+ (\gamma-1)\, M_e^2 f'\{2\dot{f}' + 2\tau(\dot{f}f''-f'\dot{f}') - ff'' - f'''\} = 0 \;. \end{aligned} \tag{2.68}$$

Using (2.65) in this expression, we have

$$2\dot{g}(1-\tau f') - g'' - g'(f-2\tau\dot{f})-(\gamma-1)\, M_e^2 f''^2 = 0 \;. \tag{2.69}$$

Equations (2.65) and (2.69) have been obtained and solved by Rodkiewicz and Reshotko [1].

Since g, the non-dimensional local enthalpy ratio, is given by

$$g = \frac{h}{h_e} = \frac{\theta}{\theta_e} = \frac{\theta}{\theta_0}\left(1 + \frac{\gamma-1}{2} M_e^2\right), \tag{2.70}$$

we may use (2.67) to obtain an expression for static temperature θ referred to the free stream stagnation temperature θ_0,

$$\frac{\theta}{\theta_0} = S(\xi,\eta,\tau) - f'^2(\xi,\eta,\tau) \ . \tag{2.71}$$

Relation (2.71) is valid only if the Mach number is large.

CHAPTER III

SOLUTION OF THE GOVERNING EQUATIONS

3.1 Finding a Solution

The problem formulated in Chapter II requires the solution of a highly nonlinear third order parabolic system of partial differential equations in which there are three independent variables. Such a complex problem does not willingly yield a closed form solution, so a suitable numerical technique must be looked for.

3.2 The Numerical Procedure

The method used to solve equations (2.52) and (2.53) is based on ideas originated by Hartree and Womersley [26] and developed by Smith and Clutter [27] for the solution of the incompressible laminar boundary layer equations. Later investigations extended the technique to the compressible laminar boundary layer [28]. The method was also found capable of handling binary dissociating mixtures and transverse curvature effects[29 , 30,31]. Many of the details of the procedure can be found in these references.

3.2a Description of Method

The fundamental idea for the method of solution is that of replacing the streamwise ξ-derivatives, which are only of first order, by forward finite difference approximations. The unsteady τ-derivatives,

also of first order, are handled the same way. The remainder of the equation is left unchanged and as a consequence the equation is converted to an ordinary differential equation which is solved over and over again as one proceeds along the ξ- and τ-directions. The use of forward differences requires that solutions be known at upstream locations.

While marching in the ξ-direction, the value of τ is held constant. When ξ has become sufficiently large, τ is increased by $\Delta\tau$, the unsteady derivatives are correspondingly updated, and the marching procedure is repeated starting at $\xi = 0$. This sequence is continued until solutions have been obtained over the desired range of ξ and τ, e.g. for $0 \leq \xi \leq 10$ and $0 \leq \tau \leq 1$.

3.2b Modifying the Equations for Numerical Solution

During the development of the numerical method, Smith and Clutter found that roundoff errors arising during the integrations could be reduced by a redefinition of the dependant variables. We introduce

$$\begin{aligned} F &= f - \eta \,, \\ F' &= f' - 1 \,, \\ F'' &= f'' \,, \\ F''' &= f''' \,, \end{aligned} \tag{3.1}$$

and

$$G = S - 1 ,$$

$$G' = S' , \qquad (3.2)$$

$$G'' = S'' .$$

These variables go to zero at the edge of the boundary layer whereas the original dependant variables f' and S go to unity.

By using the Euler transformation [21] as applied to the time variable,

$$\zeta = \frac{\tau}{\tau+1} , \qquad (3.3)$$

we map the region $0 \leq \tau \leq \infty$ into the more easily handled region $0 \leq \zeta \leq 1$; the result is an improvement in the convergence rate of the numerical method.

Applying expressions (3.1), (3.2), and (3.3) to equations (2.52) and (2.53), we obtain equations which are well suited to a numerical solution. After slight rearranging, these equations become

$$F''' = - (F+\eta)F'' - \xi[F'' \frac{\partial F}{\partial \xi} - (F'+1) \frac{\partial F'}{\partial \xi}]$$

$$+ 2(1-\zeta)^2 \frac{\partial F'}{\partial \zeta} - 2\zeta(1-\zeta)[(F'+1) \frac{\partial F'}{\partial \zeta} - F'' \frac{\partial F}{\partial \zeta}] \qquad (3.4)$$

and

$$G'' = -(F+\eta)G' - \xi[G' \frac{\partial F}{\partial \xi} - (F'+1) \frac{\partial G}{\partial \xi}]$$

$$+ 2(1-\zeta)^2 \frac{\partial G}{\partial \zeta} - 2\zeta(1-\zeta)[(F'+1) \frac{\partial G}{\partial \zeta} - G' \frac{\partial F}{\partial \zeta}] \; . \qquad (3.5)$$

3.2c The Associated Boundary Conditions

It is of course necessary to modify the boundary and initial conditions in accordance with the previous section. Hence the relations (3.1), (3.2), and (3.3) are used in expressions (2.54) through (2.62) to yield

1. At $\eta = \eta_e$,

$$F'(\xi,\eta_e,\zeta) = 0 \; , \qquad (3.6)$$

$$G(\xi,\eta_e,\zeta) = 0 \; . \qquad (3.7)$$

2. At $\eta = 0$,

$$F'(\xi,0,\zeta) = \frac{A}{\xi} F''(\xi,0,\zeta)-1 \; , \; \xi > \delta\xi \; , \; \zeta \neq 0 \; ; \qquad (3.8)$$

$$F(\xi,0,\zeta) = 0 \; , \qquad (3.9)$$

$$G(\xi,0,\zeta) = \frac{\theta_p}{\theta_0} + \frac{B}{\xi} G'(\xi,0,\zeta)$$

$$+ \frac{(A^2-2AB)}{\xi^2} F''^2(\xi,0,\zeta)-1 \; , \; \xi > \delta\xi \; , \; \zeta \neq 0 \; . \qquad (3.10)$$

The restrictions on (3.8) and (3.10) are required in order to eliminate inconsistencies in the boundary conditions at $\xi = 0$ and $\zeta = 0$.

For the case of no wall heat transfer, we have

$$G'(\xi,0,\zeta)_{ad} = 0 \ . \qquad (3.11)$$

3. At $\zeta = 0$,

$$F'(\xi,\eta,0) = F_1'(\xi_1,\eta_1)\,\frac{U_{e1}}{U_{e3}} \qquad (3.12)$$

and

$$G(\xi,\eta,0) = \frac{H_2}{H_{e3}} - 1 \ . \qquad (3.13)$$

We further specify [32] H_2 such that

$$\frac{\partial G}{\partial \zeta}(\xi,\eta,0) = 0 \ . \qquad (3.14)$$

4. At $\xi = 0$,

$$F'(0,\eta,\tau) = 0 \ , \qquad (3.15)$$

$$G(0,\eta,\tau) = 0 \ . \qquad (3.16)$$

Thus for the third order momentum equation we have three

conditions on η and one each on ξ and ζ. The energy equation, which is of second order, has two conditions on η, one on ξ, and one on ζ. The problem is therefore completely specified.

3.3 The Finite Difference Formulation

The general form of the forward finite difference approximation to the ξ- and ζ-derivatives is as follows:

When only one upstream solution is known, derivatives are replaced by the two-point formula

$$\left.\frac{\partial D}{\partial I}\right|_{I=I_n} \approx \frac{D_n - D_{n-1}}{I_n - I_{n-1}} . \tag{3.17}$$

If two or more upstream solutions are known, a more accurate three-point formula is used,

$$\begin{aligned}\left.\frac{\partial D}{\partial I}\right|_{I=I_n} \approx\ & \left[\frac{1}{I_n - I_{n-1}} + \frac{1}{I_n - I_{n-2}}\right] D_n \\ & - \left[\frac{I_n - I_{n-2}}{(I_n - I_{n-1})(I_{n-1} - I_{n-2})}\right] D_{n-1} \\ & + \left[\frac{I_n - I_{n-1}}{(I_n - I_{n-2})(I_{n-1} - I_{n-2})}\right] D_{n-2} .\end{aligned} \tag{3.18}$$

These are general equations in which D_n is the value of the dependent variable D at the n^{th} I-location I_n, where I is the independant variable of differentiation. The above forms do not require a constant

stepsize in I.

3.3a Errors in the Finite-Difference Approximations

The errors introduced by using expressions (3.17) and (3.18) are [27], respectively

$$E_2 = \frac{I_n - I_{n-1}}{2} \left.\frac{\partial^2 D}{\partial I^2}\right|_{I=I_n} \tag{3.19}$$

and

$$E_3 = \frac{(I_n - I_{n-1})(I_n - I_{n-2})}{6} \left.\frac{\partial^3 D}{\partial I^3}\right|_{I=I_n} . \tag{3.20}$$

With the Hartree-Womersley technique the most important error parameter is not the stepsize ΔI_n, but is instead the ratio $I_n/\Delta I_n$. Smith and Clutter have devoted much effort to studying the effects of this parameter, and they have learned that sensitivity problems and error propagation is reduced if the value of $I_n/\Delta I_n$ does not exceed 25. This means that to have the same accuracy in the solution at all stations the stepsize ΔI_n must increase with increasing I_n. The ability to change the stepsize ΔI_n in a prescribed manner has been built into the computer program.

3.3b The Approximated Momentum Equation

There are six forms of the momentum equation which are needed for the numerical solution. Each form is used at a different point in the (ξ,ζ) plane. Figure 1 illustrates their use.

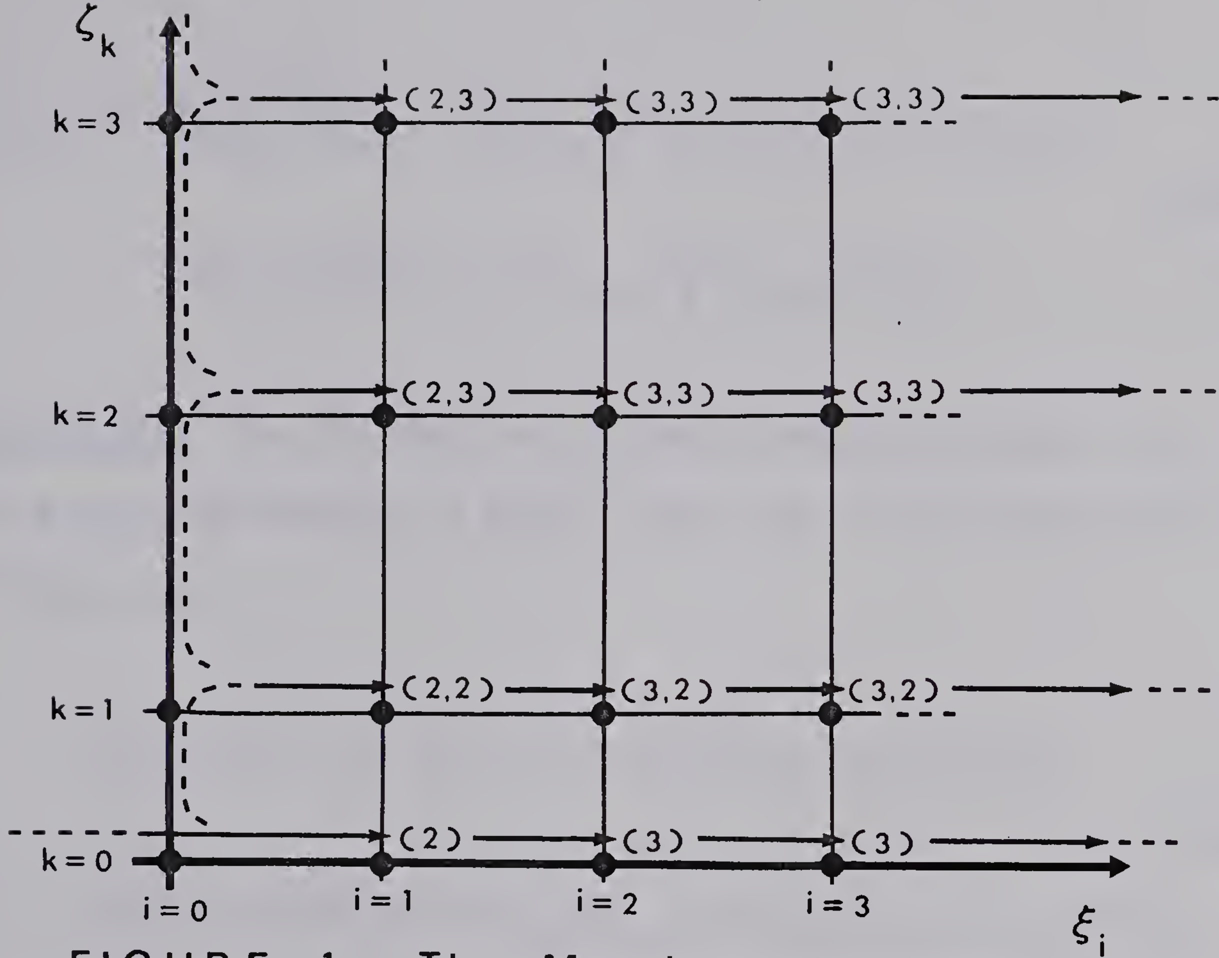

FIGURE 1. The Marching Scheme Used In The Numerical Solution

If we assume that $\Delta I_n = (I_n - I_{n-1}) = (I_{n-1} - I_{n-2})$, then the equations may be written as follows:

<u>Equation (2)F:</u> This is a steady-state equation with 2-point derivatives in ξ, and is used only at the points $(\xi_{i=1}, \eta, \zeta_{k\leq 0})$. It has the form

$$F'''_{i,k} = -(F_{i,k}+\eta)F''_{i,k} - \xi_i[F''_{i,k}(F_{i,k}-F_{i-1,k})-(F'_{i,k}+1)(F'_{i,k}-F'_{i-1,k})]/\Delta\xi_i. \tag{3.21}$$

<u>Equation (3)F:</u> This is also a steady-state equation, but it uses 3-point derivatives in ξ. Used only at the points $(\xi_{i\geq 2}, \eta, \zeta_{k\leq 0})$, it has the form

$$F'''_{i,k} = -(F_{i,k}+\eta)F''_{i,k} - \xi_i[F''_{i,k}(\tfrac{3}{2}F_{i,k}-2F_{i-1,k}+\tfrac{1}{2}F_{i-2,k})$$

$$-(F'_{i,k}+1)(\tfrac{3}{2}F'_{i,k}-2F'_{i-1,k}+\tfrac{1}{2}F'_{i-2,k})]/\Delta\xi_i\ . \tag{3.22}$$

Equation (2,2)F: This is the first of the unsteady equations. It employs 2-point derivatives in both ξ and ζ, and is used only at the points $(\xi_{i=1},\eta,\zeta_{k=1})$.

$$F'''_{i,k} - \text{R.H.S. of (2)F.} = 2(1-\zeta_k)^2(F'_{i,k}-F'_{i,k-1})/\Delta\zeta_k$$

$$-2\zeta_k(1-\zeta_k)[(F'_{i,k}+1)(F'_{i,k}-F'_{i,k-1})-F''_{i,k}(F_{i,k}-F_{i,k-1})]/\Delta\zeta_k\ . \tag{3.23}$$

Equation (3,2)F: Here we use a 3-point derivative in ξ and a 2-point derivative in ζ at the points $(\xi_{i\geq 2},\eta,\zeta_{k=1})$.

$$F'''_{i,k} = \text{R.H.S. of (3)F.} + \text{R.H.S. of (2,2)F.} \tag{3.24}$$

Equation (2,3)F: Since there are now solutions available at two previous ζ-positions, a 3-point derivative in ζ is used. The 2-point ξ-derivative is retained. This equation is used at the points $(\xi_{i=1},\eta,\zeta_{k\geq 2})$ and has the form

$$F'''_{i,k} - \text{R.H.S. of (2)F.} = 2(1-\zeta_k)^2(\tfrac{3}{2}F'_{i,k}-2F'_{i,k-1}+\tfrac{1}{2}F'_{i,k-2})/\Delta\zeta_k$$

$$- 2\zeta_k(1-\zeta_k)[(F'_{i,k}+1)(\frac{3}{2}F'_{i,k}-2F'_{i,k-1}+\frac{1}{2}F'_{i,k-2})$$

$$- F''_{i,k}(\frac{3}{2}F_{i,k}-2F_{i,k-1}+\frac{1}{2}F_{i,k-2})]/\Delta\zeta_k \ . \qquad (3.25)$$

<u>Equation (3,3)F:</u> This equation is used over the interior of the (ξ,ζ) plane. Used at points $(\xi_{i\geq 2},\eta,\zeta_{k\geq 2})$, it employs 3-point derivatives in both ξ and ζ.

$$F'''_{i,k} = \text{R.H.S. of (3)F.} + \text{R.H.S. of (2,3)F.} \qquad (3.26)$$

3.3c The Approximated Energy Equation

The energy equation also requires six forms for use at various points in the (ξ,ζ) plane. They are

<u>Equation (2)G:</u> For points $(\xi_{i=1},\eta,\zeta_{k\leq 0})$,

$$G''_{i,k} = -(F_{i,k}+\eta)G'_{i,k}-\xi_i[G'_{i,k}(F_{i,k}-F_{i-1,k})-(F'_{i,k}+1)(G_{i,k}-G_{i-1,k})]/\Delta\xi_i . \qquad (3.27)$$

<u>Equation (3)G:</u> For points $(\xi_{i\geq 2},\eta,\zeta_{k\leq 0})$,

$$G''_{i,k} = -(F_{i,k}+\eta)G'_{i,k}-\xi_i[G'_{i,k}(\frac{3}{2}F_{i,k}-2F_{i-1,k}+\frac{1}{2}F_{i-2,k}) \qquad (3.28)$$

$$- (F'_{i,k}+1)(\frac{3}{2}G_{i,k}-2G_{i-1,k}+\frac{1}{2}G_{i-2,k})]/\Delta\xi_i \ .$$

Equation (2,2)G: For points $(\xi_{i=1},\eta,\zeta_{k=1})$,

$$G''_{i,k} - \text{R.H.S. of (2)G.} = 2(1-\zeta_k)^2(G_{i,k}-G_{i,k-1})/\Delta\zeta_k$$

$$- 2\zeta_k(1-\zeta_k)[(F'_{i,k}+1)(G_{i,k}-G_{i,k-1})-G'_{i,k}(F_{i,k}-F_{i,k-1})]/\Delta\zeta_k \quad . \tag{3.29}$$

Equation (3,2)G: For points $(\xi_{i\geq 2},\eta,\zeta_{k=1})$,

$$G''_{i,k} = \text{R.H.S. of (3)G.} + \text{R.H.S. of (2,2)G} \; . \tag{3.30}$$

Equation (2,3)G: For points $(\xi_{i=1},\eta,\zeta_{k\geq 2})$,

$$G''_{i,k} - \text{R.H.S. of (2)G.} = 2(1-\zeta_k)^2(\tfrac{3}{2}\, G_{i,k}-2G_{i,k-1}+ \tfrac{1}{2}\, G_{i,k-2})/\Delta\zeta_k$$

$$- 2\zeta_k(1-\zeta_k)[(F'_{i,k}+1)(\tfrac{3}{2}\, G_{i,k}-2G_{i,k-1}+ \tfrac{1}{2}\, G_{i,k-2})$$

$$- G'_{i,k}(\tfrac{3}{2}\, F_{i,k}-2F_{i,k-1}+ \tfrac{1}{2}\, F_{i,k-2})]/\Delta\zeta_k \; . \tag{3.31}$$

Equation (3,3)G: For points $(\xi_{i\geq 2},\eta,\zeta_{k\geq 2})$,

$$G''_{i,k} = \text{R.H.S. of (3)G.} + \text{R.H.S. of (2,3)G.} \tag{3.32}$$

3.4 Integration in the ξ- and ζ- Directions

Computations are begun by finding the solution to the initial steady-state problem. The equations to be solved are the first two

in section 3.3c. Equation (3.21) is used for i=1 and equation (3.22) for $i \geq 2$. The boundary conditions for these equations are

$$
\begin{aligned}
F_{1_i}(\xi_{1_i},0) &= 0 , \\
F'_{1_i}(\xi_{1_i},0) &= \frac{A}{\xi_{1_i}} F''_{1_i}(\xi_{1_i},0)-1 \; ; \; \xi_{1_i} > \delta\xi_1 , \\
F'_{1_i}(\xi_{1_i},\eta_{1_e}) &= 0 , \\
F''_{1_i}(\xi_{1_i},\eta_{1_e}) &= 0 .
\end{aligned}
\tag{3.33}
$$

The extra condition (on F'') is required because the value of η_e is a function of ξ, and should not be fixed a priori. The assumption is also made that free-stream conditions exist at the leading edge of the plate. Thus

$$
\begin{aligned}
F_{1_{i=0}}(0,\eta_1) &= 0 , \\
F'_{1_{i=0}}(0,\eta_1) &= 0 , \\
F''_{1_{i=0}}(0,\eta_1) &= 0 .
\end{aligned}
\tag{3.34}
$$

As a consequence it is not necessary to compute a solution at $\xi = 0$.

The equations are solved by marching in the increasing ξ-direction, starting with $\xi_{i=1} = 0.1$. Six steps are taken with

$\Delta\xi_i = \xi_i - \xi_{i-1} = 0.1$, followed by seven steps with $\Delta\xi_i = 0.2$, then six steps where $\Delta\xi_i = 0.4$. At further ξ-positions the stepsize $\Delta\xi = 0.8$ is used. Marching ceases when the solution obtained is within ten percent of the asymptotic zero-slip solution obtained by Cohen and Reshotko [33]. This point is reached at approximately $\xi = 10$.

At each ξ-position the unknown boundary condition $F_i''(\xi_i,0)$ must be found. This is done in the following manner:

(1) A guess is made for the value of $F_i''(\xi_i,0)$. When i=1, the value 0.1 is used. For $i \geq 2$, the value found at the $(i-1)^{th}$ station is used as the initial guess.

(2) The momentum equation is integrated over the η-variable until a value of η is reached such that either $F_i'(\xi_i,\eta) \geq 0$ or $F_i''(\xi_i,\eta) \leq 0$. If integration is terminated by the first condition, the guessed value of $F_i''(\xi_i,0)$ was too large. If termination was due to the second condition, $F_i''(\xi_i,0)$ was guessed too small. Based on this information, new guesses are made at intervals of $\Delta F_i''(\xi_i,0) = 0.1$.

(3) Once the true value of $F_i''(\xi_i,0)$ has been bracketed by the above method, a bisection search technique is used to obtain a refined estimate of $F_i''(\xi_i,0)$, and the momentum equation is integrated again with the more accurate starting value.

(4) The search-integrate procedure is repeated until the conditions $F_i'(\xi_i,\eta_e) = 0$ and $F_i''(\xi_i,\eta_e) = 0$ are satisfied to four places. This requires a high degree of accuracy in the value of $F_i''(\xi_i,0)$ due to error propagation in the integration technique.

The sensitivity to errors is inherent in the Hartree-Womersly formulation.

After the steady-state momentum equation has been integrated out to $\xi = 10$, boundary condition (3.12) is used to obtain the velocity profiles which exist immediately after acceleration takes place (i.e. at $\zeta = 0$). Thus we calculate

$$F_{i,k=0}(\xi_i,\eta,0) = \left(\frac{U_{e1}}{U_{e3}}\right)^{1/2} F_{1_i}\left(\left[\frac{U_{e1}}{U_{e3}}\right]^{1/2} \xi_i, \left[\frac{U_{e3}}{U_{e1}}\right]^{1/2} \eta\right) ,$$

$$F'_{i,k=0}(\xi_i,\eta,0) = \frac{U_{e1}}{U_{e3}} F'_{1_i}\left(\left[\frac{U_{e1}}{U_{e3}}\right]^{1/2} \xi_i , \left[\frac{U_{e3}}{U_{31}}\right]^{1/2} \eta\right) , \quad (3.35)$$

$$F''_{i,k=0}(\xi_i,\eta,0) = \left(\frac{U_{e1}}{U_{e3}}\right)^{3/2} F''_{1_i}\left(\left[\frac{U_{e1}}{U_{e3}}\right]^{1/2} \xi_i, \left[\frac{U_{e3}}{U_{e1}}\right]^{1/2} \eta\right) ,$$

where U_{e3} is arbitrarily given the value 1.01 U_{e1}.

We now have the solutions which are assumed to exist at $\zeta = 0$. The next step is to find the solutions at non-zero values of ζ. The method used is similar to that which was applied for non-zero values of ξ, in that finite-difference approximations replace the ζ-derivatives. The valid equations are given in section 3.3c. Equation (3.23) is used for the case when (i=1,k=1), and equation (3.24) is used whenever we have ($i \geq 2$, $k = 1$). Expression (3.25) is used to obtain solutions at the points ($i=1$, $k \geq 2$), and equation (3.26) represents the most general case where ($i \geq 2$, $k \geq 2$). Thus equations (3.23) through (3.26) are used to march slowly in the increasing ζ-direction. The stepsize

$\Delta\zeta_k$ is varied as follows: $\Delta\zeta_{k=1} = 0.05$ and $\Delta\zeta_{k\geq 2} = 0.1$. At each ζ-level the momentum equation is integrated over $0 < \xi \leq 10$ using the marching technique described previously.

The distance that can be marched in the timewise direction is limited by the singularity which exists at $\zeta_c = 0.5$(i.e. $\tau=1$). Discovered by Stewartson [34] and later confirmed by many others [35,36,1], the singularity causes the method of solution to become unstable in the region [18] of

$$\zeta_c = \frac{(2-\beta)}{2(F'+1)[(\frac{\gamma}{\gamma-1})\ \beta+1] + (2-\beta)} \quad . \tag{3.36}$$

At this value of ζ a discontinuity appears [37] in the f'distribution, and the present method becomes incapable of proceeding any further in the ζ-direction.

Once the solution to the momentum equation is known for all $0 \leq \xi \leq 10$ and $0 \leq \zeta \leq \zeta_c$, the energy equation can be solved with the same method.

3.5 Integration of the Ordinary Differential Equations

Integration of the O.D.E.'s is carried out by means of a standard integrating subroutine contained in IBM's Scientific Subroutine Package [38]. This subroutine, called DHPCG, uses Hamming's modified predictor-corrector method with double-precision arithmetic to integrate a system of first-order O.D.E's with given initial values. It is therefore necessary to replace the third-order momentum equation

with three coupled first-order equations, and the second-order energy equation with two coupled first-order equations. This is done quite simply by defining five new functions such that

$$F1 = F ,$$

$$F2 = \frac{dF1}{d\eta} = F' , \tag{3.37}$$

$$F3 = \frac{dF2}{d\eta} = F'' ,$$

and

$$G1 = G ,$$

$$G2 = \frac{dG1}{d\eta} = G' . \tag{3.38}$$

This form is readily accepted by subroutine DHPCG. Details of the procedure can be found in Ref. [38,39]. It should be mentioned here that one of the advantages of this subroutine is that it will maintain a specified accuracy by varying the stepsize in η.

3.6 Errors in the Smith-Cutter Technique

Since the ordinary differential equations must be solved by an initial-value procedure, a "shooting" technique is used in order to determine the initial values (unknown wall boundary conditions) F_w'' and G_w' which satisfy the known free stream conditions. The correct solution is obtained by what is essentially a trial-and-error method; the amount

of error introduced is dependent on the degree of accuracy with which the free-stream conditions are satisfied. In the present case an accuracy of four places was specified. Thus the maximum error introduced by the shooting technique is in the fifth place and occurs at the edge of the boundary layer. The error at smaller values of η must be less since about ten-place accuracy in the wall conditions is required to obtain four-place accuracy at η_e.

Errors are also introduced when the value of η_e is obtained. As the edge of the boundary layer is approached, subroutine DHPCG finds that a stepsize of $\Delta\eta = 0.1$ is small enough to satisfy the error requirements imposed on the integration procedure. Thus the value of η_e is accurate only to one decimal place. However, the results show that changing the value of η_e by the amount $\Delta\eta$ affects the solutions in the fifth place only.

Inaccuracies due to round-off errors can sometimes be significant. Double precision computations using the very fine stepsize $\Delta\eta = 0.01$ were compared with solutions obtained using the stepsize $\Delta\eta = 0.1$; agreement existed in the tenth place.

Smith and Clutter found that the boundary layer equations as formulated herein would become extremely sensitive to the wall conditions if the ratio $I_n/\Delta I_n$ exceeded 25. While it is possible to control this parameter in the present case, there is an additional complication which arises as a result of the unsteady terms $\partial F'/\partial\zeta$ and $\partial G/\partial\zeta$. These terms have the form

$$\frac{1}{\Delta I_n}(\ldots)$$

whereas all other finite-difference terms in the equations can be written as

$$\frac{I_n}{\Delta I_n}(\ldots) .$$

Even if $\zeta_k/\Delta\zeta_k$ is kept small (unity, for example) it is quite possible for $1/\Delta\zeta_k$ to be one hundred times as large. The result is that the unsteady term dominates the differential equation in such a way that an incorrect value of F_w'' or G_w' produces errors which increase exponentially [27] as the edge of the boundary layer is approached. While various numerical methods [40,41,42,43] have been developed to eliminate this problem, a less exotic procedure known as the Extended Trajectory of Integration (ETI) technique is used. The problem is not removed, it is merely circumvented by breaking the integration into a number of shorter steps over which the errors have insufficient time to get out of hand. This method is described very well in Ref. [31].

Smith and Clutter [44] use a surprisingly simple but nevertheless excellent method to investigate the question of instability of solution in the streamwise direction. They do this by solving a problem which has a similar solution. A similar flow is one of the few for which accurate solutions exist. Because the flow is similar, the solution should be identical at all stations and error growth is just the change in solutions.

In the present case, similarity is obtained by employing the no-slip condition at the wall. Then the steady-state solution becomes identical with that obtained by Cohen and Reshotko [33]. When both types of solutions are compared, agreement to four places is obtained, even at large values of ξ. Figure 2a shows errors as a function of ξ at the $\eta = 0.1$ level. Figure 2b shows error growth across the boundary layer at the station $\xi = 4.4$. These curves are typical and show that the errors resulting from Smith and Clutter's technique are negligible for all practical purposes. Since no similar solutions exist in the timewise direction, it is impossible to do a similar check on instability along the ζ-axis. We can only hope that the accuracy there is as good.

It is felt that the solutions presented herein are accurate to at least four places. This upper bound on the accuracy is reached at the edge of the boundary layer; accuracy should improve at lower values of η.

CHAPTER IV

APPLICATION OF THE SOLUTIONS

4.1 Shear Stress at the Wall

The shear stress at the wall has the usual form

$$\Phi_w = \mu_w \frac{\partial u}{\partial y}\Big|_{y=0} \tag{4.1}$$

Application of the transformations of Chapter II yields

$$\Phi_w = \mu_w \frac{\rho_w}{\rho_{ref}} \left(\frac{a_e}{a_{ref}}\right)^2 \sqrt{\frac{(m+1)\ V_e}{2\nu_{ref}\ X}}\ V_e\ f_w''. \tag{4.2}$$

The local skin friction coefficient may be defined as

$$C_f = \frac{\Phi_w}{\frac{1}{2}\ \rho_w\ U_e^2} \tag{4.3}$$

Combining (4.2) and (4.3) yields

$$\begin{aligned} C_f &= 2\ \frac{\mu_w}{\rho_{ref}} \left(\frac{a_e}{a_{ref}}\right)^2 \sqrt{\frac{(m+1)\ V_e}{2\nu_{ref}\ X}}\ \frac{V_e}{U_e^2}\ f_w'' \\ &= 2\ \frac{1}{\sqrt{\frac{\pi\gamma'}{2}}}\ \frac{a_w}{U_e}\ \frac{1}{\xi}\ f_w'' , \end{aligned} \tag{4.4}$$

or

$$C_f = 2 \frac{1}{\sqrt{\frac{\pi\gamma}{2}}} \frac{1}{M_w} \frac{u_w}{U_e} \frac{1}{\xi} f''_w . \qquad (4.5)$$

We now introduce a Knudsen number based on fluid properties evaluated at the wall,

$$K_n = \frac{\lambda_w}{\nu_w/u_w} , \qquad (4.6)$$

which may be written as

$$K_n = \sqrt{\frac{\pi\gamma}{2}} \, M_w . \qquad (4.7)$$

Combining this relation with equation (4.5), the result is

$$\frac{C_f K_n}{2} = \frac{1}{\xi} \, f'_w \, f''_w , \qquad (4.8)$$

or

$$\frac{C_f K_n}{2} = \frac{1}{A} \, f'^2_w . \qquad (4.9)$$

4.2 Heat Transfer at the Wall

Aroesty [13] notes that the energy transferred to a wall from a slightly rarefied gas in motion is given by

$$Q_w = q_w + u_w \, \Phi_w , \qquad (4.10)$$

where q_w, the Fourier heat conduction term, has the usual form

$$q_w = k_w \frac{\partial\theta}{\partial y}\Big|_{y=0} \quad , \tag{4.11}$$

and $u_w \Phi_w$, the slip work term, is found from

$$u_w\Phi_w = \mu_w u_w \frac{\partial u}{\partial y}\Big|_{y=0} \quad . \tag{4.12}$$

Since $Pr = C_p\mu/k$, we have

$$k_w = \frac{C_p\mu_w}{Pr} \quad . \tag{4.13}$$

Then

$$Q_w = \frac{C_p\mu_w}{Pr}\frac{\partial\theta}{\partial y}\Big|_{y=0} + \mu_w\, u_w \frac{\partial u}{\partial y}\Big|_{y=0} \quad . \tag{4.14}$$

This equation is transformed as was equation (4.1). The result is

$$Q_w = \mu_w \frac{\rho_w}{\rho_{ref}}\frac{a_e}{a_{ref}}\left[\frac{(m+1)}{2\nu_{ref}}\frac{V_e}{X}\right]^{1/2}\left[\frac{C_p\theta_{ref}}{Pr} S'_w - \frac{B}{\xi} f''^{2}_w\left(2\frac{C_p\theta_{ref}}{Pr} - a_e^2 M_e^2\right)\right] \quad . \tag{4.15}$$

We assume that $Pr = 1$ and note that if the stagnation values are used as the reference conditions,

$$C_p\theta_{ref} = C_p\theta_0 = H_e \tag{4.16}$$

and
$$a_e^2 M_e^2 = U_e^2 . \tag{4.17}$$

Then (4.15) may be written as

$$Q_w = H_e \mu_w \frac{\rho_w}{\rho_{ref}} \frac{a_e}{a_{ref}} \left[\frac{(m+1)}{2\nu_{ref}} \frac{V_e}{X}\right]^{1/2} \left[S_w' - \frac{B}{\xi} f_w''^2 \left(2 - \frac{U_e^2}{H_e}\right)\right] . \tag{4.18}$$

We recall that equation (2.45) was

$$\frac{U_e^2}{H_e} = \frac{(\gamma-1) M_e^2}{1 + \frac{(\gamma-1)}{2} M_e^2}$$

For large Mach numbers an approximate form is

$$\frac{U_e^2}{H_e} = 2 . \tag{4.19}$$

Combining (4.18) and (4.19) we have

$$Q_w = H_e \mu_w \frac{\rho_w}{\rho_{ref}} \frac{a_e}{a_{ref}} \left[\frac{(m+1)}{2\nu_{ref}} \frac{V_e}{X}\right]^{1/2} S_w' . \tag{4.20}$$

Choosing the stagnation values for reference, and using expressions (A.3), (A.5), and (A.18), equation (4.20) becomes

$$Q_w = H_e \rho_w a_w \sqrt{\frac{2}{\pi\gamma}} \frac{1}{\xi} S_w' . \tag{4.21}$$

Defining Nusselt number as

$$\mathrm{Nu} = \frac{Q_w \lambda_w}{k_w(\theta_{ref} - \theta_w)} \tag{4.22}$$

we obtain, after substituting for Q_w from (4.21),

$$\mathrm{Nu} = \frac{H_e \rho_w a_w \sqrt{\frac{2}{\pi\gamma}}\, Pr\, \lambda_w\, S_w'}{C_p \mu_w \left(1 - \frac{\theta_w}{\theta_{ref}}\right) \theta_{ref}\, \xi}$$

$$= \frac{a_w}{\nu_w} \frac{1}{\sqrt{\frac{\pi\gamma}{2}}} \frac{1}{\xi} \frac{S_w' \lambda_w}{(1 - S_w + f_w'^2)} ,$$

or

$$\mathrm{Nu} = \frac{1}{\xi} \frac{S_w'}{(1 - S_w + f_w'^2)} \quad . \tag{4.23}$$

The numerical results for the transient contributions to the shear stress and heat transfer at the wall are presented in Appendix D.

CHAPTER V

RESULTS AND CONCLUSIONS

5.1 Discussion of the Results

The transient behavior of the boundary layer on a flat plate following an impulsive acceleration of the surrounding fluid has been analysed. Some of the effects of low density have been included by using boundary conditions which admit a slip velocity and a temperature jump at the wall. The equations governing the boundary layer, after being rewritten for a coordinate system suggested by the boundary conditions, have been solved by a numerical finite-difference technique developed by Smith and Clutter [27]. In all cases the unsteady disturbances were initiated by an impulsive 1% increase in the free stream velocity.

Several special cases of the momentum equation (2.49) have been examined by other workers. Berard [25] studied the problem of weak interaction (β=0) steady-state slip flow. Figures 3a and 3b show some of his experimental data superimposed on theoretical curves from the present work. The best agreement was obtained when curves corresponding to a tangential momentum accommodation coefficient σ = 0.6 were compared with Berard's data with pressure corrections. This rather low value for σ corresponds to reflection from an aluminum plate covered with a very thin film of oil. The oil film, which was

observed by Berard, is more "specular" than the metal surface and hence lowers the value of the accommodation coefficient.

Aroesty [13] worked on the problem of strong interaction steady-state slip flow. He found that an approximate solution for the wall shear in adiabatic slip flow was

$$f''_w = f''_{B_w} - \varepsilon\beta \tag{5.1}$$

or, in terms of the present theory,

$$f''_w = 0.765 - \frac{\beta}{\xi} . \tag{5.2}$$

In Figure 3c this solution is compared with that from the present analysis. Agreement is seen to be good except in the region of small ξ, where Aroesty's approximate solution predicts a separated flow.

It is seen that the present steady-state theory produces solutions for both the strong and weak interaction problems which compare well with those in the literature. This lends support to the various assumptions made during the course of the analysis, and provides a strong base from which the assault on the unsteady problem can be launched.

Examination of the curves in Appendix D shows that the unsteady problem has been solved for only a few steps in either the ξ- or ζ- directions. The decision to limit the computed solutions was

based on purely economic reasons. About three minutes of computer time were required to obtain the solution at each point in the ξ-ζ plane. The original intention to solve at 25 ξ-steps (to ξ = 10) and 10 ζ-steps (to ζ=0.5) would have required well over 12 hours of non-stop computation. It is felt that, since the method of solution has been developed, the solutions at larger values of ξ and ζ can be obtained with relatively little difficulty.

The transient contributions to the velocity and shear functions f' and f'' are plotted in Figures 4, 5a and 5b. The transients are observed to increase with ξ, a result which is in keeping with the fact that the leading-edge profile does not change with time. Since the transients are referred to the steady slip solution, the ξ-dependence of the transients is not related to the ξ-dependence of the slip velocity, but is instead a leading-edge effect which should disappear as ξ becomes large. Thus there will be some point on the plate beyond which the transient contribution to the velocity and shear will cease to be a function of ξ, with the result that the transients far downstream (ξ>10), should they be computed, will perhaps approach the results of Rodkiewicz and Reshotko [1]. This trend is evidenced by the increasing curvature of the slight hump in the $\Delta f'$ curves at low values of η as ξ is increased.

Figures 6a and 6b serve to illustrate the time-dependence of the drag function. The drag function is seen to decrease monotonically with both ξ and ζ, varying directly with the square of the slip velocity. The behavior of the drag near the leading edge is

open to question, since it depends so much on the assumed shape of the leading-edge profile. However, at the leading edge the drag coefficient C_f defined by equation (4.3) can, by use of the slip condition, be reduced to

$$C_f = \frac{2}{M_e\sqrt{\frac{\pi\gamma}{2}}} , \tag{5.3}$$

which is the value corresponding to free molecular flow [9]. Furthermore, examination of the expression

$$C_f = \frac{2}{M_e\sqrt{\frac{\pi\gamma}{2}}}\{f_w'[S_w + (\tfrac{\gamma-1}{2})M_e^2(S_w - f_w'^2)]^{1/2}\} , \tag{5.4}$$

obtained from equations (4.9) and (B.19), shows that for large Mach numbers the factor in curly brackets is greater than unity over at least a small part of the range of ξ. For large ξ, however, the low value of f_w' forces the factor to become less than one. Qualitatively this behavior agrees precisely with that described by Kogan [8] wherein he states that at large Mach numbers, as the Knudsen number decreases (increasing ξ), the drag and the heat transfer first increase above their free-molecule values, reach a maximum, and then decrease to values corresponding to a continuous medium. A detailed study of this behavior will require that solutions be obtained over the entire range of ξ.

The transient contributions to the total enthalpy function

and the total enthalpy gradient are presented in Figures 7 and 8. The transient behavior is qualitatively the same as was observed for the velocity and shear functions, with the addition of a slight dependence on the temperature of the plate. The indications are that for a cold plate the heat transfer in the interior of the unsteady slipping boundary layer is less sensitive to the passage of time than is the interior region of the boundary layer on a hot plate. Conversely, the upper region of the cold boundary layer seems to react more quickly to the passage of time than does the upper region of a hot boundary layer. This behavior contrasts with that observed by Gupta and Rodkiewicz [18] in which they found that throughout its thickness a cold non-slipping boundary layer reacted more slowly than would a hot zero-slip boundary layer. They attributed this effect to the fact that, for the hot wall, the boundary-layer density is less than that for the cold wall, rendering the hot-wall boundary layer more susceptible to thermodynamic changes than the cold-wall boundary layer. Their curves, obtained for the case of strong interactions, show that the sensitivity difference between the hot and the cold boundary layer decreases with β. A linear extrapolation of their data indicates that for the case of weak interactions (β=0) a cold boundary layer should be more sensitive (have larger temporal gradients) than a hot boundary layer. This agrees with the present findings, at least in the outer regions of the boundary layer. One possible explanation for the insensitivity at the interior of a cold slipping boundary layer is that the gas in the neighborhood of the wall (e.g. $0.1 < \eta < 0.5$) is com-

paratively dense so that its temperature is insensitive to small fluctuations in the heat transfer across the region. The absolute viscosity and the heat conduction coefficient, both being direct functions of temperature, are low and are relatively time independent. Thus the heat transfer across the interior region remains fairly constant, and most of the transient readjustments take place in the upper levels of the boundary layer. The situation immediately adjacent to the wall ($\eta < 0.1$) is different. There the gas temperature is low enough for the unsteady variation to be a significant fraction of the steady state value. The result is that the heat transfer at the surface of a cold plate is more sensitive to time than is the heat transfer at the surface of a hot plate. This sensitivity shows up as a more rapid approach to the steady-state value, as illustrated in Figures 9a and 9b.

The steady-state heat transfer distribution, plotted in Figure 9c, is seen to exhibit a monotonic decrease with increasing ξ, but shows no sign of the behavior predicted by Kogan [8]. This indicates that the classical slip boundary conditions can not be relied upon to provide accurate drag and heat transfer estimates close to the leading edge. Therefore the drag profile given in Figure 6b should be used with caution, as the matching with free-stream conditions at the leading edge is merely a coincidence, and is not a result of the accuracy of the boundary conditions.

Since the primary purpose of the present work was to explore the transient behavior of a slipping boundary layer, it was not felt

necessary to include plots of the steady-state velocity and temperature profiles. Also, solutions to the steady slip problem already exist in the literature. However, for reasons of completeness this and other data have been included in tabular form, and may be found in Appendix E.

Reproductions of the computer programs used to solve the energy and momentum equations are presented in Appendix F with the hope that they may be of some assistance to the reader.

5.2 Concluding Remarks

The present results may be used to obtain the history of the drag and heat transfer at the surface of a flat plate in slipping high-speed flow. Although the data presented is quite limited, the computer programs are such that very few changes are required in order to produce solutions which extend almost to the zero-slip region.

Once the solutions for the entire range of ξ have been completed it would be desirable to obtain the solution to cases in which there are strong interactions. This would of course require that the pressure gradient parameter β be a known function of x which satisfies the conditions $\beta(\text{small } \xi) = (\gamma-1)/\gamma$ and $\beta(\text{large } \xi) = 0$. Similar investigations for other airfoil profiles such as a wedge, circular cone etc. should also be of interest.

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APPENDIX A

TRANSFORMATION OF THE RAREFIED GAS BOUNDARY CONDITIONS

A.1 Velocity at the Wall

The usual boundary condition on the velocity at the wall of a rarefied gas flowing at high speed over a surface is known as the "slip velocity boundary condition". Reference [13] shows it may be written as

$$u(x,0,t) = A\lambda_w \frac{\partial u(x,0,t)}{\partial y} , \tag{A.1}$$

where A, a constant for a given gas-surface combination, is computed from

$$A = \frac{2-\sigma}{\sigma} , \tag{A.2}$$

and λ_w, a function of x and t, is the mean free path of a gas molecule adjacent to the surface. It is given by

$$\lambda_w = \frac{\mu_w}{\rho_w a_w} \sqrt{\frac{\pi\gamma}{2}} . \tag{A.3}$$

If the transformations of equation (2.33), Chapter II, are applied to equation (A.1), then the transformed boundary condition is

$$u(X,0,T) = A\lambda_w \frac{\rho_w}{\rho_{ref}} \frac{a_e}{a_{ref}} \frac{\partial u(X,0,T)}{\partial Y} . \tag{A.4}$$

When we define

$$L \equiv L(X,T) = \frac{\lambda_w \rho_w a_e}{\lambda_{ref}\rho_{ref}a_{ref}} , \tag{A.5}$$

we may write

$$u(X,0,T) = A\lambda_{ref}L \frac{\partial u(X,0,T)}{\partial Y} . \tag{A.6}$$

At first glance it might appear that L depends on θ_w and thus creates coupling between the velocity and temperature at the wall. Appendix B shows why this need not be so.

We introduce the incomplete coordinate transformations

$$\xi = \text{some unknown function of } X \text{ and } T , \tag{A.7}$$

$$\eta = \left[\frac{(m+1)V_e}{2\nu_{ref}X}\right]^{1/2} Y , \tag{A.8}$$

$$\tau = \frac{V_e T}{X} . \tag{A.9}$$

We also make use of the new stream function f defined by

$$\psi = \left[\frac{2\nu_{ref}X\ V_e}{(m+1)}\right]^{1/2} f(\xi,\eta,\tau) . \tag{A.10}$$

Now

$$u(X,Y,T) = \frac{a_e}{a_{ref}} \frac{\partial\psi}{\partial Y} \tag{A.11}$$

and

$$\frac{\partial\psi}{\partial Y} = V_e \; f'(\xi,\eta,\tau) \; , \tag{A.12}$$

or

$$\frac{\partial\psi}{\partial Y} = a_{ref} \; M_e \; f'(\xi,\eta,\tau) \; . \tag{A.13}$$

Therefore

$$u(X,Y,T) = a_e M_e f'(\xi,\eta,\tau) \; . \tag{A.14}$$

Differentiating with respect to Y,

$$\frac{\partial u}{\partial Y} = a_e \; M_e \; [\frac{(m+1) \; V_e}{2\nu_{ref} \; X}]^{1/2} \; f''(\xi,\eta,\tau) \; . \tag{A.15}$$

Substituting (A.14) and (A.15) into (A.6), the boundary condition becomes

$$a_e M_e f'(\xi,0,\tau) = A\lambda_{ref} L a_e M_e \; [\frac{(m+1) \; V_e}{2\nu_{ref} \; X}]^{1/2} \; f''(\xi,0,\tau) \; , \tag{A.16}$$

which reduces to

$$f'(\xi,0,\tau) = A\lambda_{ref}L \left[\frac{(m+1)\, V_e}{2\nu_{ref}\, X}\right]^{1/2} f''(\xi,0,\tau) \ . \qquad (A.17)$$

We note that if a semi-similar solution was desired, wherein $f \equiv f(\eta,\tau)$, the above slip boundary condition would be functionally inconsistent because of the X- and T-dependence on the right hand side.

Since the independent variable ξ is still unspecified, functional consistency will be obtained if we let

$$\xi = \frac{1}{\lambda_{ref}L} \left[\frac{2\nu_{ref}\, X}{(m+1)\, V_e}\right]^{1/2} \ . \qquad (A.18)$$

Appendix B shows that this definition of ξ produces an independent variable that depends only on X.

From (A.18), the boundary condition becomes

$$f'(\xi,0,\tau) = \frac{A}{\xi}\, f''(\xi,0,\tau) \ . \qquad (A.19)$$

A.2 Temperature at the Wall

Reference [5] gives us the usual form for the boundary condition on the gas temperature at a surface immersed in high speed slightly rarefied flow. It is

$$\theta(x,0,t) = \theta_p + B\lambda_w \frac{\partial\theta(x,0,t)}{\partial y} \ , \qquad (A.20)$$

where θ_p is the plate temperature, which is constant, and B is a constant given by

$$B = \left(\frac{2\gamma}{\gamma+1}\right)\left(\frac{1}{Pr}\right)\left(\frac{2-\alpha}{\alpha}\right) . \tag{A.21}$$

When equation (A.20) is transformed as was (A.1), the result is

$$\theta(X,0,T) = \theta_p + B\lambda_w \frac{\rho_w}{\rho_{ref}} \frac{a_e}{a_{ref}} \frac{\partial\theta(X,0,T)}{\partial Y}. \tag{A.22}$$

Introducing L,

$$\theta(X,0,T) = \theta_p + B\lambda_{ref}L \frac{\partial\theta(X,0,T)}{\partial Y} . \tag{A.23}$$

Dividing through by θ_0 and applying (2.71), we obtain

$$S(\xi,0,\tau)-f'^2(\xi,0,\tau) = \frac{\theta_p}{\theta_0} + B\lambda_{ref}L \frac{\partial}{\partial Y}[S(\xi,0,\tau)-f'^2(\xi,0,\tau)] . \tag{A.24}$$

We now make use of the coordinate transformations found in section A.1. Thus

$$S(\xi,0,\tau) - f'^2(\xi,0,\tau) = \frac{\theta_p}{\theta_0} + B\lambda_{ref}L \left[\frac{(m+1)}{2\nu_{ref}} \frac{V_e}{X}\right]^{1/2} [S'(\xi,0,\tau)$$

$$- 2f'(\xi,0,\tau)\, f''(\xi,0,\tau)] . \tag{A.25}$$

Substituting (A.18) and (A.19) into the above equation, then rearranging, yields the boundary condition representing a temperature jump at the wall,

$$S(\xi,0,\tau) = \frac{\theta_p}{\theta_0} + \frac{B}{\xi} S'(\xi,0,\tau) + \frac{(A^2-2AB)}{\xi^2} f''^2(\xi,0,\tau) \ . \qquad (A.26)$$

A.3 Condition for an Adiabatic Wall

From Chapter IV we know that

$$Q_w = H_e \rho_w a_w \sqrt{\frac{2}{\pi\gamma}} \frac{1}{\xi} S'(\xi,0,\tau) \ . \qquad (A.27)$$

For an insulated wall, we must have $Q_w = 0$, and therefore

$$S'(\xi,0,\tau)_{ad} = 0 \ . \qquad (A.28)$$

APPENDIX B

EVALUATION OF THE QUANTITY L, THE PARAMETER ξ, AND THE KNUDSEN NUMBER

B.1 An Examination of L

We wish to examine the quantity L more closely. Since

$$\lambda = \frac{\mu}{\rho a}\sqrt{\frac{\pi\gamma}{2}} \tag{B.1}$$

and

$$L = \frac{\lambda_w \rho_w a_e}{\lambda_{ref}\rho_{ref}a_{ref}} , \tag{B.2}$$

we have

$$L = \frac{\dfrac{\mu_w}{\rho_w a_w}\sqrt{\dfrac{\pi\gamma}{2}}\ \rho_w\ a_e}{\dfrac{\mu_{ref}}{\rho_{ref}a_{ref}}\sqrt{\dfrac{\pi\gamma}{2}}\ \rho_{ref}a_{ref}} ,$$

or

$$L = \frac{\mu_w a_e}{\mu_{ref}a_w} . \tag{B.3}$$

Hayes and Probstein [45] note that at high temperatures

$$\mu \propto \theta^{1/2} \quad . \tag{B.4}$$

Thus

$$L = \frac{\theta_w^{1/2} \sqrt{\gamma R \theta_e}}{\theta_{ref}^{1/2} \sqrt{\gamma R \theta_w}}$$

$$= \sqrt{\frac{\theta_e}{\theta_{ref}}} \, , \tag{B.5}$$

or, using stagnation conditions as the reference,

$$L = [1 + \frac{\gamma-1}{2} M_e^2]^{-1/2} \quad . \tag{B.6}$$

We note here that, although (B.4) is inconsistent with equation (2.20), its use is very convenient in that it renders L independent of the gas temperature at the wall. Thus L no longer depends on time, but is a function of X only. Since both (B.4) and (2.20) merely approximate the true dependence of viscosity on temperature, the inconsistency is not serious.

If the Mach number is large an approximate form for (B.6) is

$$L = \frac{1}{\sqrt{\frac{\gamma-1}{2}} \; M_e} \quad . \tag{B.7}$$

B.2 An Examination of ξ

We wish to determine how ξ depends on X. The definition of ξ, equation (2.34), suggests the form

$$\xi = a\, X^{\alpha} \; . \tag{B.8}$$

Then

$$a = \xi X^{-\alpha}$$

$$= \frac{1}{\lambda_{ref} L} \left[\frac{2\nu_{ref}}{(m+1)} \frac{X}{V_e}\right]^{1/2} X^{-\alpha} \; . \tag{B.9}$$

But

$$V_e = a_{ref} M_e = eX^m \; , \tag{B.10}$$

and from (B.7) and (B.10)

$$L = \frac{1}{\sqrt{\frac{\gamma-1}{2}} \frac{e}{a_{ref}} X^m} \; . \tag{B.11}$$

Therefore

$$a = \frac{1}{\lambda_{ref}} \frac{e}{a_{ref}} \sqrt{\frac{\gamma-1}{2}}\, X^m \left[\frac{2\nu_{ref}}{(m+1)} \frac{X}{eX^m}\right]^{1/2} X^{-\alpha}$$

$$= \frac{1}{\lambda_{ref} a_{ref}} \nu_{ref}^{1/2} e^{1/2} \sqrt{\frac{\gamma-1}{m+1}} X^{m-\frac{1}{2}(m-1)-\alpha} \quad . \tag{B.12}$$

Since we know that

$$a \neq a(X) \quad , \tag{B.13}$$

we must have

$$m - \frac{1}{2}(m-1) - \alpha = 0 \quad . \tag{B.14}$$

Using the relation

$$m = \frac{\beta}{2-\beta} \quad , \tag{B.15}$$

we have

$$\alpha = \frac{1}{2-\beta} \quad . \tag{B.16}$$

Thus ξ has the form

$$\xi = a\, X^{\frac{1}{2-\beta}} \quad . \tag{B.17}$$

This simple expression is a direct result of assumption (B.4), which had the effect of removing the time-dependence from L, and thus from ξ.

Equation (B.17) can be differentiated to yield

$$\frac{d\xi}{dX} = \frac{1}{2-\beta} \frac{\xi}{X} . \tag{B.18}$$

B.3 The Knudsen Number

From Equation (4.6) the Knudsen number is defined as

$$K_n = \frac{\lambda_w}{\nu_w/u_w} .$$

The ratio ν_w/u_w can be thought of as a local "characteristic length" for slip flow, and is used solely for convenience.

The grouping $\lambda_w/(\nu_w/u_w)$ has a rather ubiquitous nature. As written, it has the form of a Knudsen number, the significant parameter for slip flow [6]. However, it can be rearranged to $u_w\lambda_w/\nu_w$, a form of Reynolds number, or, using equation (B.1), to $\sqrt{\pi\gamma/2}\, M_w$, a direct function of the local Mach number at the wall.

An expression for the Knudsen number can be obtained in terms of the numerical solution as follows:

$$K_n = \frac{\lambda_w}{\nu_w/u_w}$$

$$= \sqrt{\frac{\pi\gamma}{2}} \frac{u_w}{a_w}$$

$$= \sqrt{\frac{\pi\gamma}{2}} \frac{u_w}{U_e} \frac{U_e}{a_e} \frac{a_e}{a_w} ,$$

or

$$K_n = M_e \sqrt{\frac{\pi\gamma}{2}}\, f'_w \left[\left(1 + \frac{\gamma-1}{2} M_e^2\right)S_w - \frac{\gamma-1}{2} M_e^2 f_w'^2\right]^{-1/2} . \quad (B.19)$$

At the leading edge this becomes

$$K_n = \sqrt{\frac{\pi\gamma}{2}}\, M_e , \quad (B.20)$$

whereas further downstream an approximate form is

$$K_n = \sqrt{\frac{\pi\gamma}{\gamma-1}}\, \frac{f'_w}{(S_w - f_w'^2)^{1/2}} \quad (B.21)$$

which is independent of the free-stream conditions so long as the Mach number is large. Essentially equation (B.19) describes how the slip effects decay as the no-slip region is approached.

APPENDIX C

CORRELATION WITH AROESTY'S WORK

We want to learn the relationship between Aroesty's perturbation parameter ε and the modified x coordinate ξ. We have [13]

$$\varepsilon = \frac{\lambda_w \rho_w U_\delta}{\sqrt{2\zeta}} , \tag{C.1}$$

and

$$\xi = \frac{1}{\lambda_{ref} L} \left[\frac{2\nu_{ref} X}{(m+1) V_e}\right]^{1/2} . \tag{C.2}$$

We note that

$$\eta_A (\eta \text{ as used by Aroesty}) = \frac{U_\delta}{\sqrt{2\zeta}} \int_0^y \rho \, dy , \tag{C.3}$$

where

$$\zeta = \int_0^x \rho_w \mu_w U_\delta \, dx \tag{C.4}$$

and

$$U_\delta = U_e . \tag{C.5}$$

In this analysis

$$\eta = \left[\frac{(m+1)\ V_e}{2\nu_{ref}\ X}\right]^{1/2} \frac{a_e}{a_{ref}} \int_0^y \frac{\rho}{\rho_{ref}}\ dy\ , \tag{C.6}$$

where

$$X = \int_0^x C\left(\frac{a_e}{a_{ref}}\right)\left(\frac{P_e}{P_{ref}}\right) dx\ . \tag{C.7}$$

Thus

$$\frac{\eta}{\eta_A} = \left[\frac{(m+1)\ V_e}{2\nu_{ref}\ X}\right]^{1/2} \frac{a_e}{a_{ref}}\ \frac{1}{\rho_{ref}}\ \frac{1}{U_e} \sqrt{2\zeta}\ . \tag{C.8}$$

Using (C.1) in (C.8), we obtain

$$\frac{\eta}{\eta_A} = \left[\frac{(m+1)\ V_e}{2\nu_{ref}\ X}\right]^{1/2} \frac{a_e}{a_{ref}}\ \frac{1}{\rho_{ref}}\ \frac{1}{U_e}\ \frac{\lambda_w\ \rho_w\ U_e}{\varepsilon}\ ,$$

$$= \lambda_w\ \frac{\rho_w}{\rho_{ref}}\ \frac{a_e}{a_{ref}} \left[\frac{(m+1)\ V_e}{2\nu_{ref}\ X}\right]^{1/2} \frac{1}{\varepsilon}\ ,$$

$$= \lambda_{ref}\ \frac{\lambda_w}{\lambda_{ref}}\ \frac{\rho_w}{\rho_{ref}}\ \frac{a_e}{a_{ref}} \left[\frac{(m+1)\ V_e}{2\nu_{ref}\ X}\right]^{1/2} \frac{1}{\varepsilon}\ ,$$

$$= \lambda_{ref} L \left[\frac{(m+1)\ V_e}{2\nu_{ref}\ X}\right]^{1/2} \frac{1}{\varepsilon}\ . \tag{C.9}$$

Employing equation (C.2), the result is

$$\frac{\eta}{\eta_A} = \frac{1}{\varepsilon\xi} \quad . \tag{C.10}$$

Returning attention to (C.8), we see that

$$\frac{\eta}{\eta_A} = \left[\frac{(m+1)\ V_e}{2\nu_{ref}\ X}\right]^{1/2} \frac{a_e}{a_{ref}} \frac{1}{\rho_{ref}} \frac{1}{U_e} \sqrt{2\zeta} \ ,$$

$$= \left[\frac{(m+1)\ \frac{a_e}{a_{ref}}\ U_e\ \left(\frac{a_e}{a_{ref}}\right)^2\ 2 \int_o^x \rho_w\ \mu_w\ U_e dx}{2\ \frac{\mu_{ref}}{\rho_{ref}}\ X\ \rho_{ref}^{\ 2}\ U_e^{\ 2}}\right]^{1/2} \tag{C.11}$$

where (C.4) and (C.5) have been used with the relations

$$V_e = \frac{a_{ref}}{a_e} U_e = a_{ref}\ M_e \ , \tag{C.12}$$

and

$$\nu_{ref} = \frac{\mu_{ref}}{\rho_{ref}} \ . \tag{C.13}$$

Then

$$\frac{\eta}{\eta_A} = \left[\frac{(m+1)\ \frac{a_e}{a_{ref}} \int_o^x \frac{\rho_w}{\rho_{ref}}\ \frac{\mu_w}{\mu_{ref}}\ U_e\ dx}{U_e\ X}\right]^{1/2}$$

$$= \left[\frac{(m+1)\int_o^x \frac{\rho_w}{\rho_{ref}} \frac{\mu_w}{\mu_{ref}} \frac{a_e}{a_{ref}} M_e \, dx}{M_e X}\right]^{1/2} . \quad (C.14)$$

We know that

$$M_e = \frac{e}{a_{ref}} x^m \quad , \quad (C.15)$$

and Reference [46] gives us

$$X = cx \frac{2\gamma}{-\frac{1}{2}(3\gamma-1) + 2\gamma} \left(\frac{P_w}{P_{ref}}\right)^{\frac{3\gamma-1}{2\gamma}} . \quad (C.16)$$

For strong interactions, it is known that

$$\frac{P_w}{P_{ref}} = \frac{P_w}{P_0} = \frac{P_e}{P_0} = kx^{-1/2} . \quad (C.17)$$

Therefore,

$$\frac{\eta}{\eta_A} = \left[\frac{(m+1)\int_o^x \frac{\rho_w}{\rho_{ref}} \frac{\mu_w}{\mu_{ref}} \frac{a_e}{a_{ref}} c^m x^m x^{-m(\frac{3\gamma-1}{4\gamma})} dx}{c^m x^m x^{-m(\frac{3\gamma-1}{4\gamma})} cx\left[\frac{2\gamma}{2\gamma - \frac{1}{2}(3\gamma-1)}\right] k^{\frac{3\gamma-1}{2\gamma}} x^{-\frac{3\gamma-1}{4\gamma}}}\right]^{1/2} . \quad (C.18)$$

Now

$$X = \int_o^x \frac{\rho_w}{\rho_{ref}} \frac{\mu_w}{\mu_{ref}} \frac{a_e}{a_{ref}} dx . \quad (C.19)$$

Therefore

$$\frac{dX}{dx} = \frac{\rho_w}{\rho_{ref}} \frac{\mu_w}{\mu_{ref}} \frac{a_e}{a_{ref}} . \tag{C.20}$$

But from (C.16) we have

$$\frac{dX}{dx} = \frac{d}{dx} [Cx^{\frac{2\gamma}{2\gamma - \frac{1}{2}(3\gamma-1)}} (kx^{-1/2})^{\frac{3\gamma-1}{4\gamma}}] ,$$

$$= C[\frac{2\gamma}{2\gamma - \frac{1}{2}(3\gamma-1)}] k^{\frac{3\gamma-1}{2\gamma}} \frac{d}{dx} (x^{1-\frac{3\gamma-1}{4\gamma}})$$

$$+ [\frac{2\gamma}{2\gamma - \frac{1}{2}(3\gamma-1)}] k^{\frac{3\gamma-1}{2\gamma}} x^{1-\frac{3\gamma-1}{4\gamma}} \frac{dC}{dx} ,$$

$$= [\frac{2\gamma}{2\gamma - \frac{1}{2}(3\gamma-1)}] k^{\frac{3\gamma-1}{2\gamma}} x^{1-\frac{3\gamma-1}{4\gamma}} [C(1-\frac{3\gamma-1}{4\gamma}) x^{-1} + \frac{dC}{dx}] . \tag{C.21}$$

Equating the right-hand side of (C.20) with the right-hand side of (C.21), and substituting the result into equation (C.18), we obtain

$$\frac{\eta}{\eta_A} = [\frac{(m+1) \int_0^x [\frac{2\gamma}{2\gamma - \frac{1}{2}(3\gamma-1)}]k^{\frac{3\gamma-1}{2\gamma}} x^{1-\frac{3\gamma-1}{4\gamma}} [C(1-\frac{3\gamma-1}{4\gamma})x^{-1} + \frac{dC}{dx}]C^m x^m x^{-m\frac{3\gamma-1}{4\gamma}} dx}{C^{m+1} x^{m(1-\frac{3\gamma-1}{4\gamma})} x^{1-\frac{3\gamma-1}{4\gamma}} [\frac{2\gamma}{2\gamma - \frac{1}{2}(3\gamma-1)}] k^{\frac{3\gamma-1}{2\gamma}}}]^{1/2} . \tag{C.22}$$

If C ≈ constant (an approximation made by Aroesty) then we can write

$$\frac{\eta}{\eta_A} = \left[\frac{(m+1)C^{m+1}\left[\frac{2\gamma}{2\gamma-\frac{1}{2}(3\gamma-1)}\right]k^{\frac{3\gamma-1}{2\gamma}}\int_0^x (1-\frac{3\gamma-1}{4\gamma})\, x^{-\frac{3\gamma-1}{4\gamma}}\, x^{m(1-\frac{3\gamma-1}{4\gamma})}\, dx}{C^{m+1}\, x^{(m+1)(1-\frac{3\gamma-1}{4\gamma})}\left[\frac{2\gamma}{2\gamma-\frac{1}{2}(3\gamma-1)}\right]k^{\frac{3\gamma-1}{2\gamma}}}\right]^{1/2},$$

$$= \left[\frac{(m+1)(1-\frac{3\gamma-1}{4\gamma})\int_0^x x^{m(1-\frac{3\gamma-1}{4\gamma})-\frac{3\gamma-1}{4\gamma}}\, dx}{x^{(m+1)(1-\frac{3\gamma-1}{4\gamma})}}\right]^{1/2} . \qquad (C.23)$$

Evaluating the integral in the numerator produces

$$\frac{\eta}{\eta_A} = \left\{\frac{(m+1)(1-\frac{3\gamma-1}{4\gamma})\, x^{m(1-\frac{3\gamma-1}{4\gamma})-\frac{3\gamma-1}{4\gamma}+1}}{[m(1-\frac{3\gamma-1}{4\gamma})-\frac{3\gamma-1}{4\gamma}+1]\, x^{(m+1)(1-\frac{3\gamma-1}{4\gamma})}}\right\}^{1/2},$$

$$= \left[\frac{(m+1)(1-\frac{3\gamma-1}{4\gamma})\, x^{(m+1)(1-\frac{3\gamma-1}{4\gamma})}}{(m+1)(1-\frac{3\gamma-1}{4\gamma})\, x^{(m+1)(1-\frac{3\gamma-1}{4\gamma})}}\right]^{1/2}$$

$$= 1 . \qquad (C.24)$$

We therefore have the simple relations

$$\eta = \eta_A , \qquad (C.25)$$

and

$$\xi = \frac{1}{\varepsilon} . \qquad (C.26)$$

APPENDIX D

CURVES FOR CHAPTERS III AND IV

- containing a graphical presentation of error growth, comparison with the work of others, and the numerical solutions.

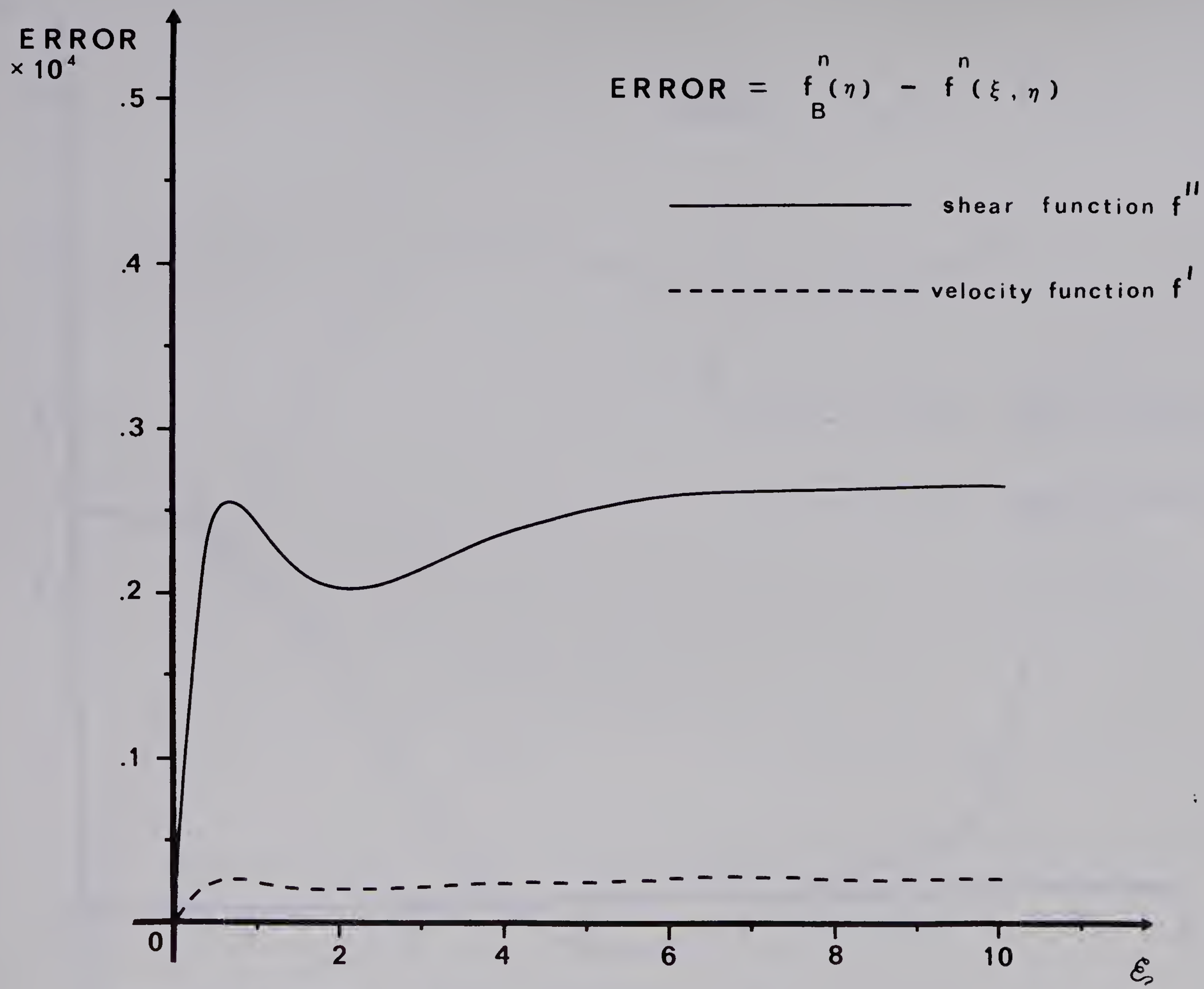

FIGURE 2a. Error Growth at the $\eta = 0.1$ Level

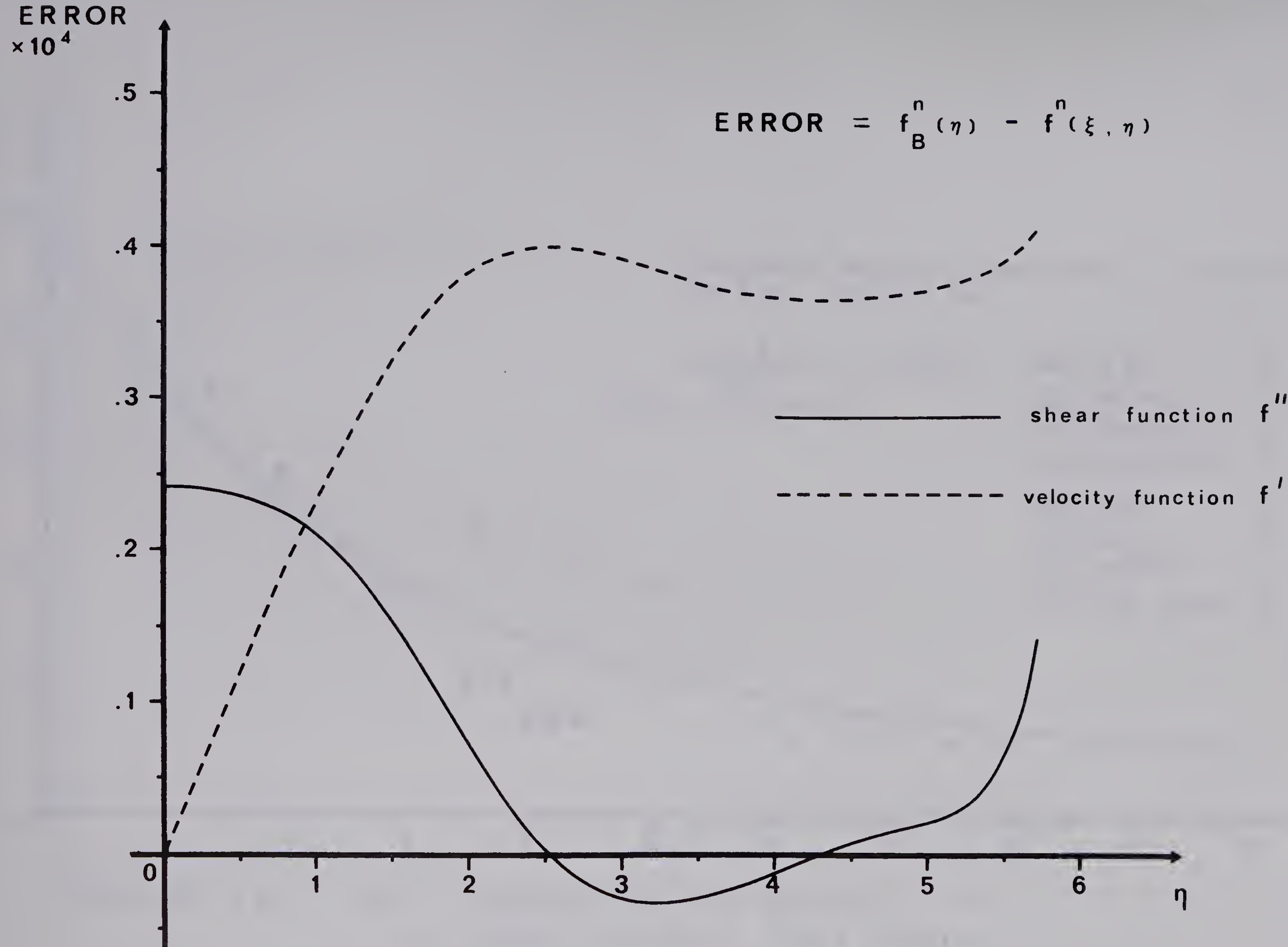

FIGURE 2b. Error Growth at $\xi = 4.4$

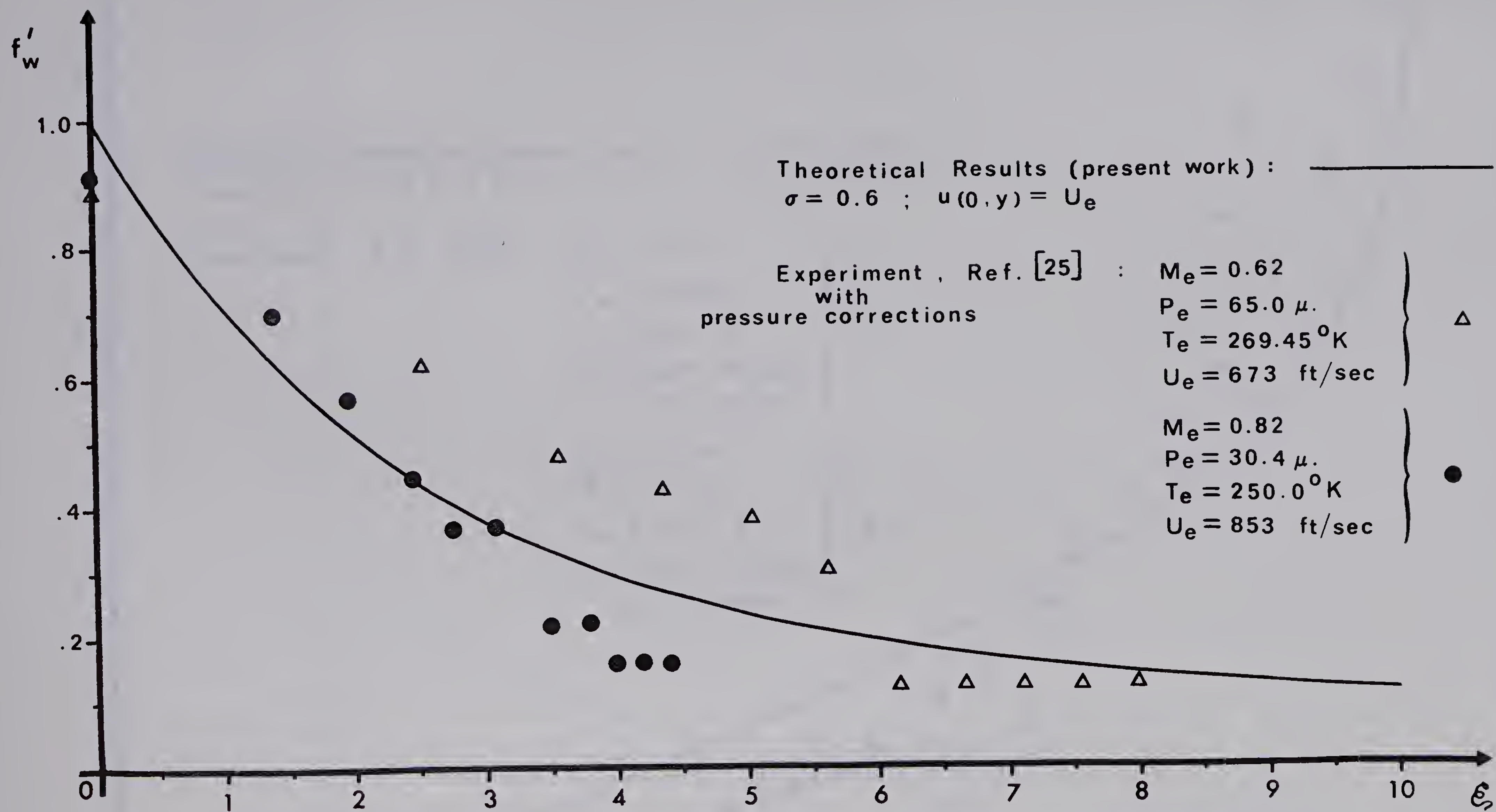

FIGURE 3a. Gas Velocity at the Surface of a Semi-Infinite Flat Plate in Steady Slip Flow

D4

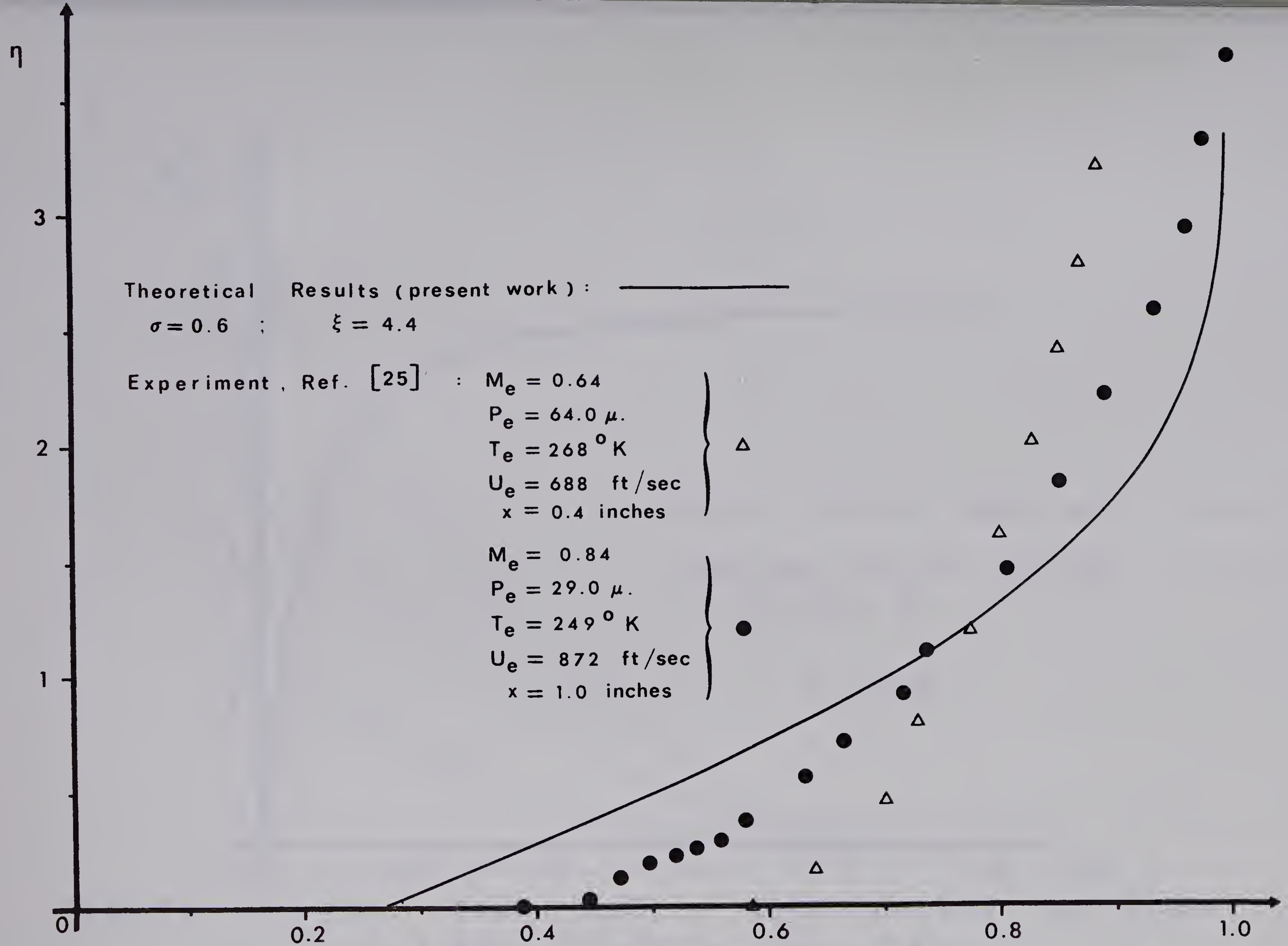

FIGURE 3b. Velocity Profile on a Semi-Infinite Flat Plate in Steady Slip Flow

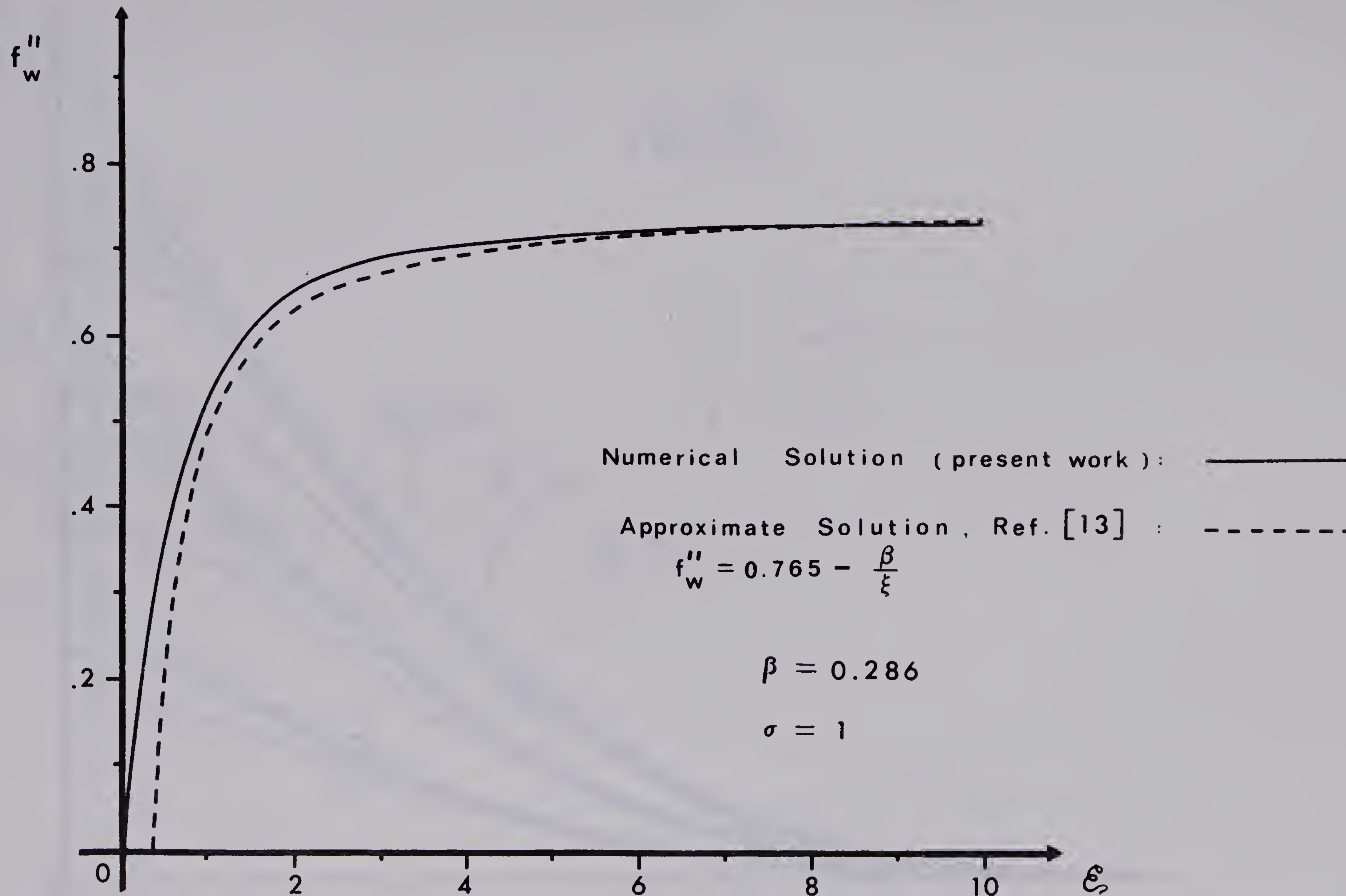

FIGURE 3c. Wall Shear Along a Semi - Infinite Flat Plate in Steady Adiabatic Slip Flow

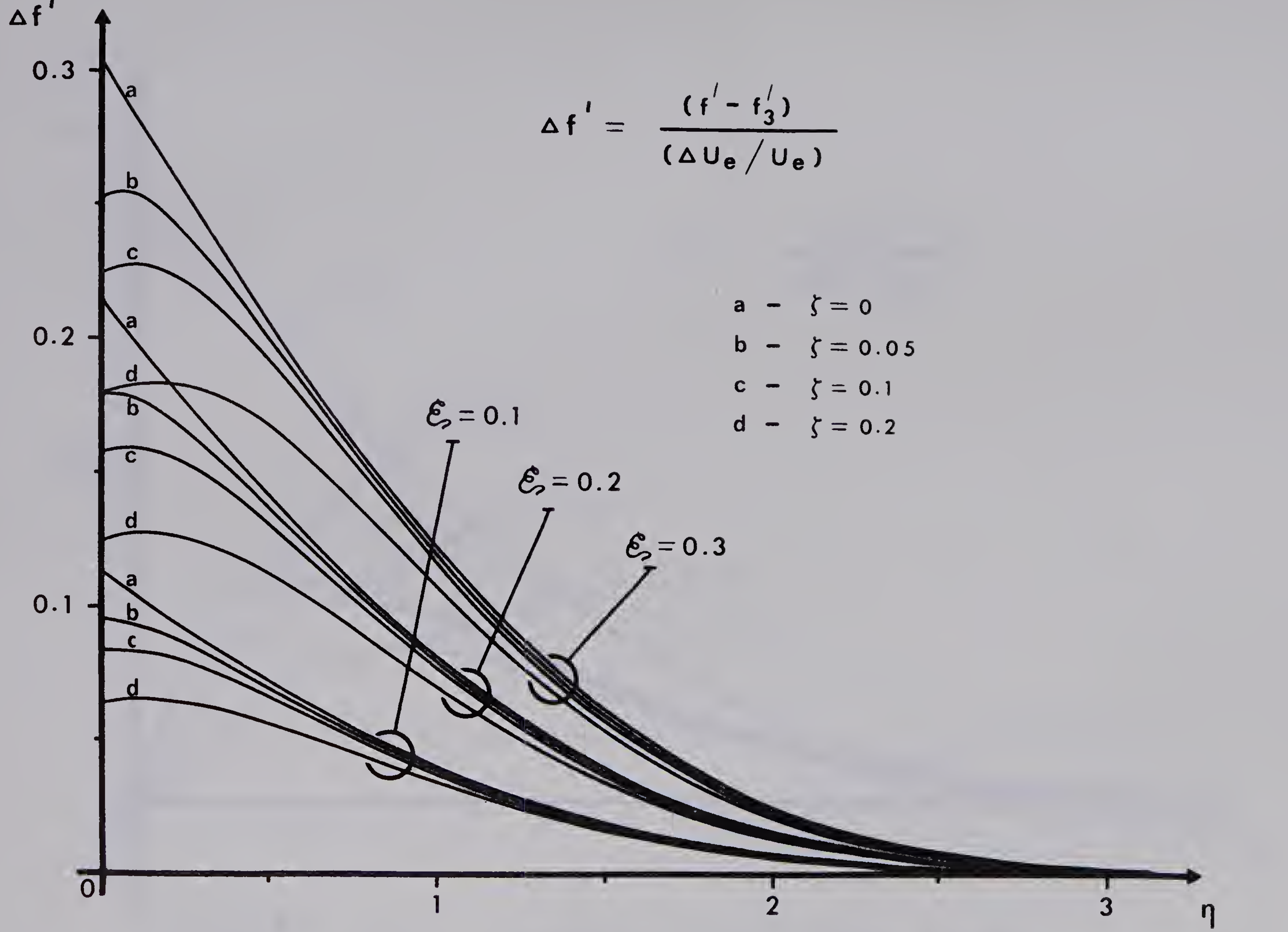

FIGURE 4. Transient Contribution to the Velocity Function

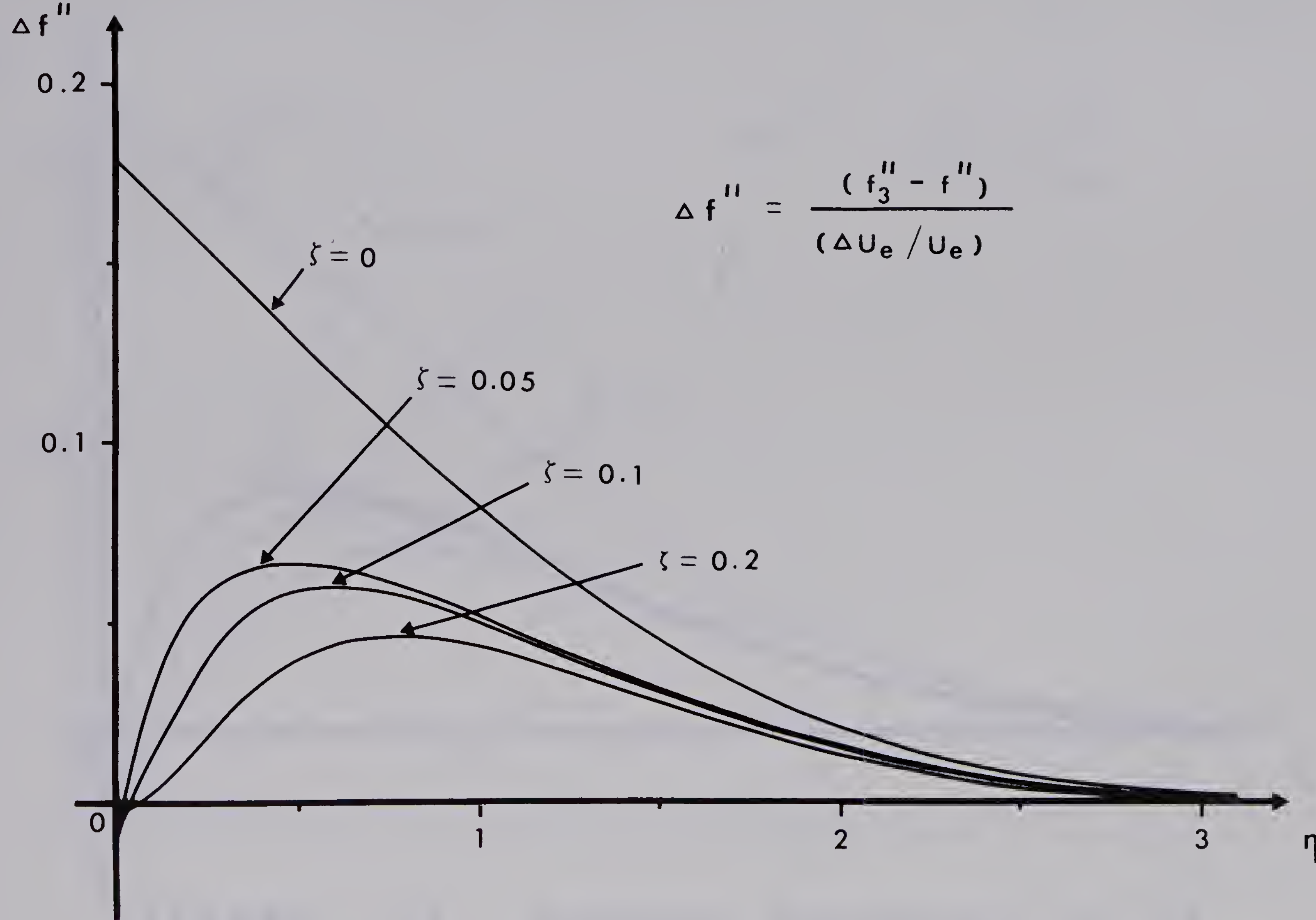

FIGURE 5a. Transient Contribution to the Shear Function at $\mathcal{E}_{\eta} = 0.1$

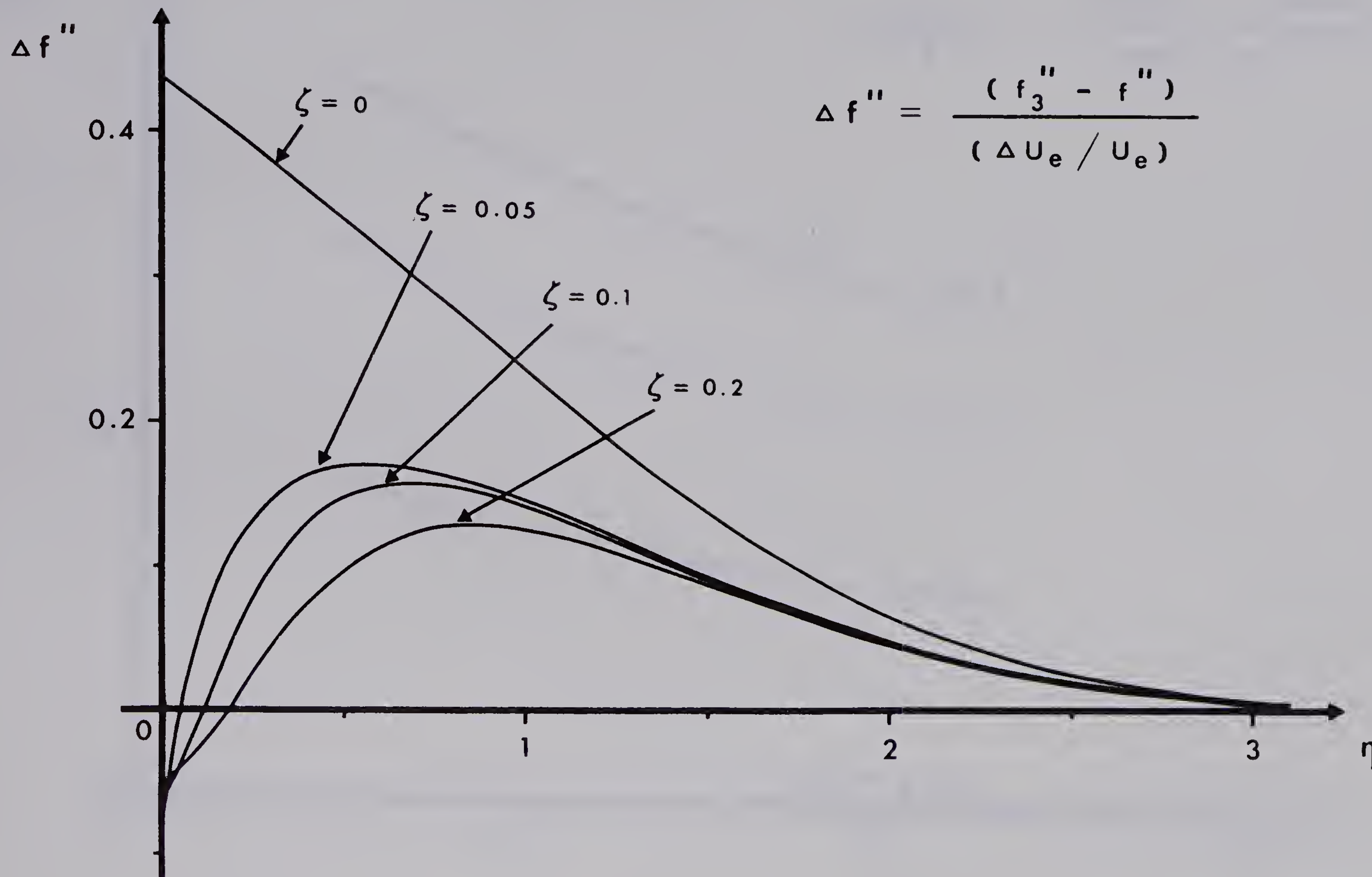

FIGURE 5b. Transient Contribution to the Shear Function at $\mathcal{E} = 0.3$

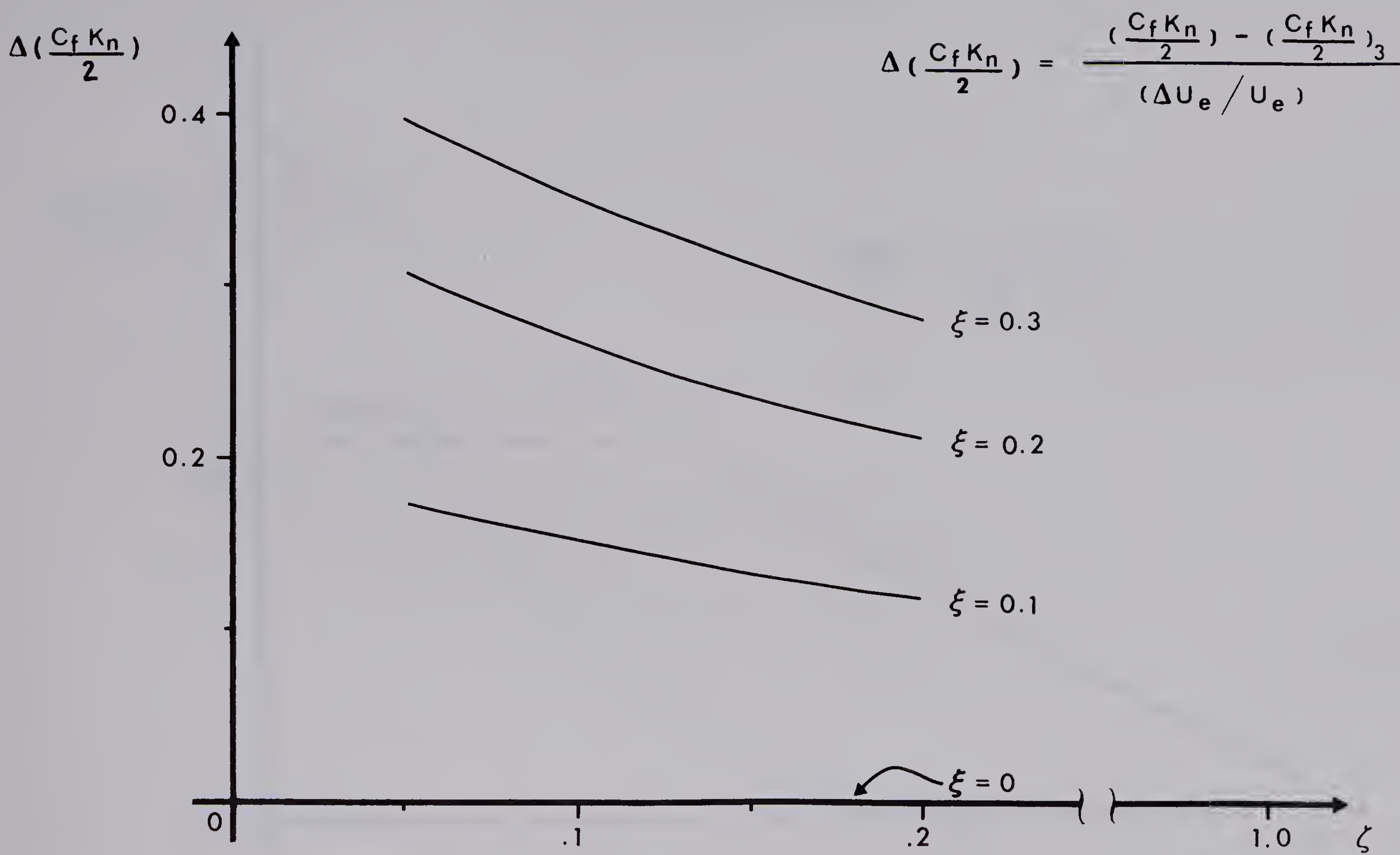

FIGURE 6a. Transient Contribution to the Drag Function

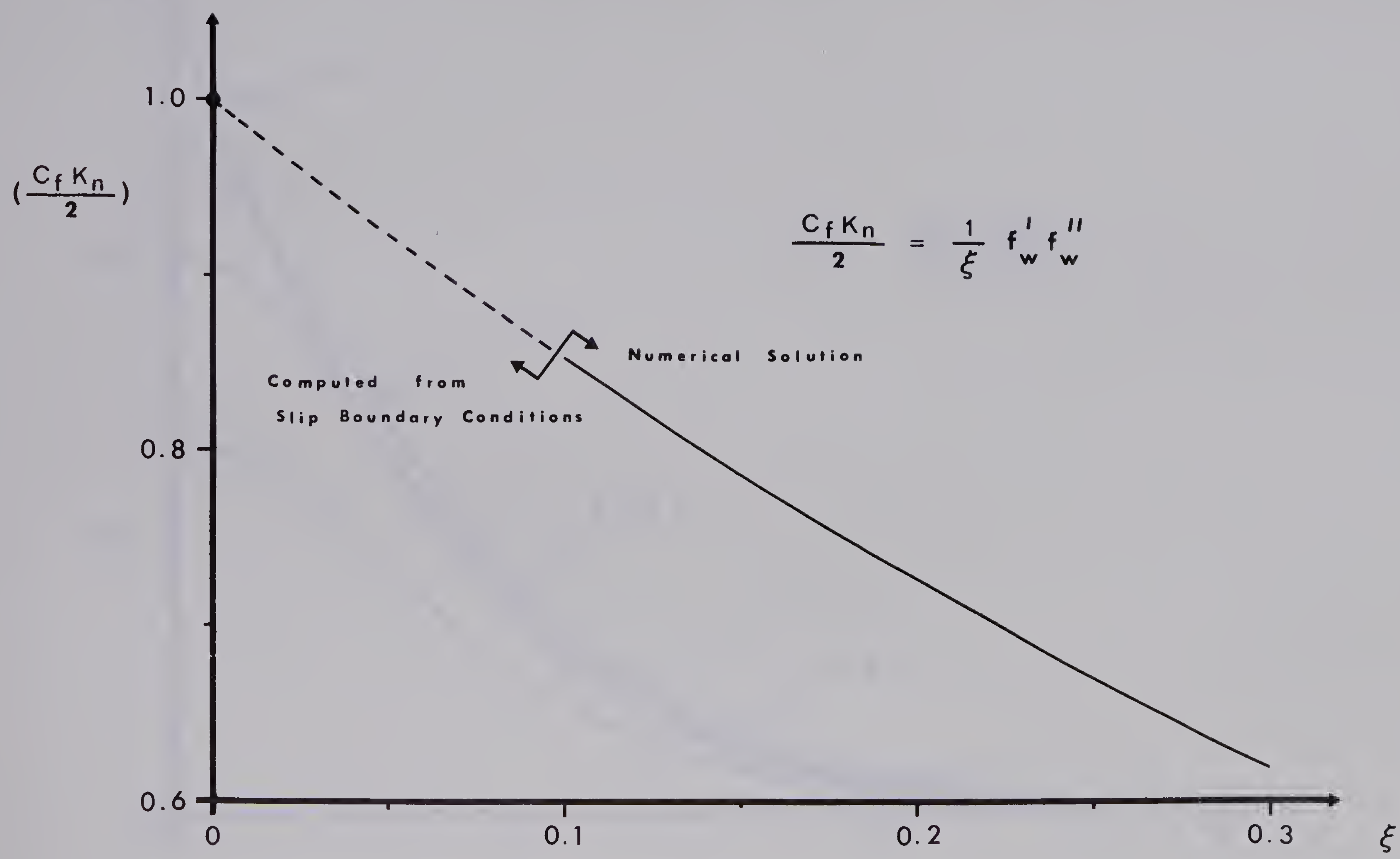

FIGURE 6b. Steady - State Distribution of the Drag Function

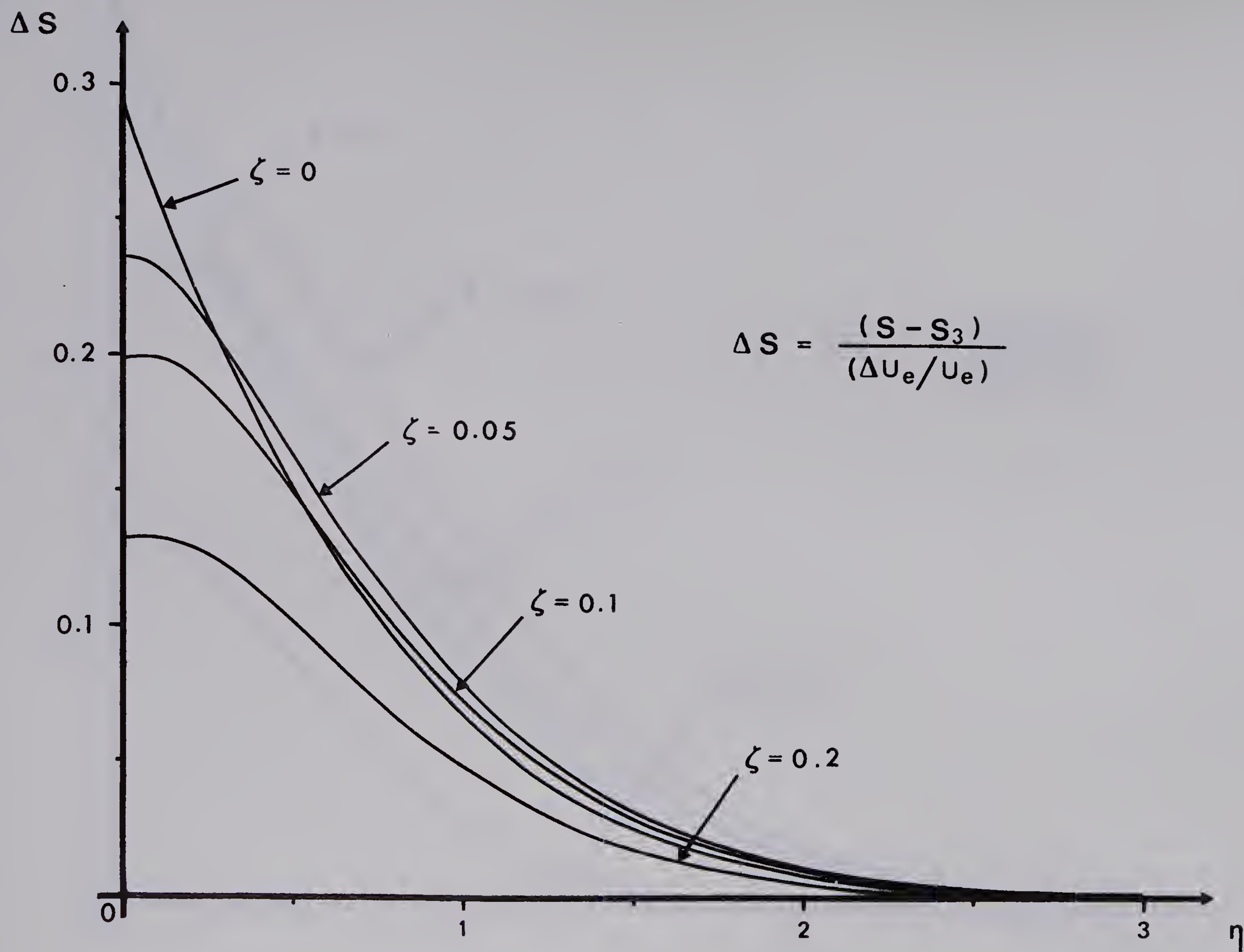

FIGURE 7a. Transient Contribution to the Total Enthalpy Function at $\xi = 0.1$ with $\theta_p/\theta_o = 0$

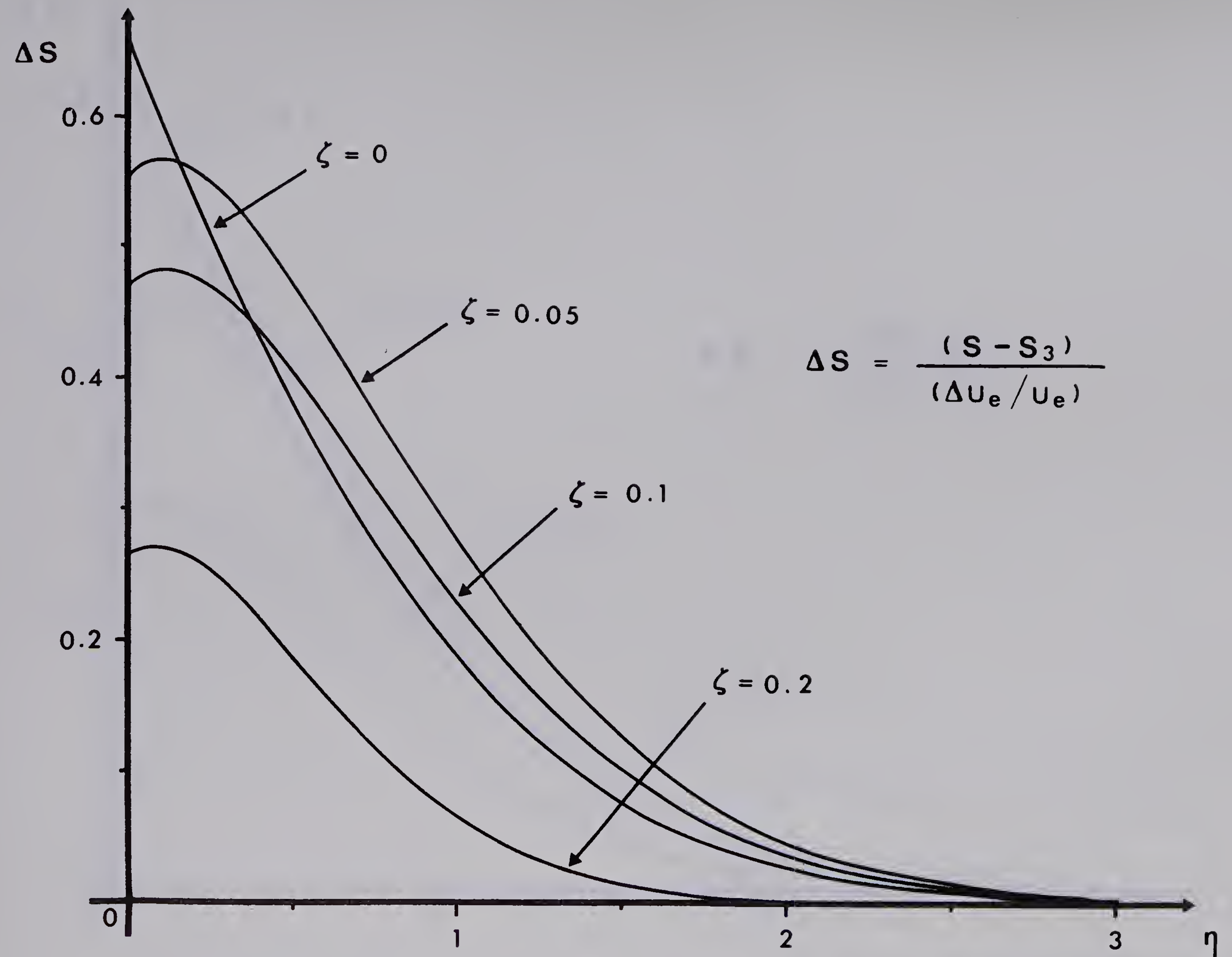

FIGURE 7b. Transient Contribution to the Total Enthalpy Function at $\xi = 0.3$ with $\theta_p / \theta_o = 0$

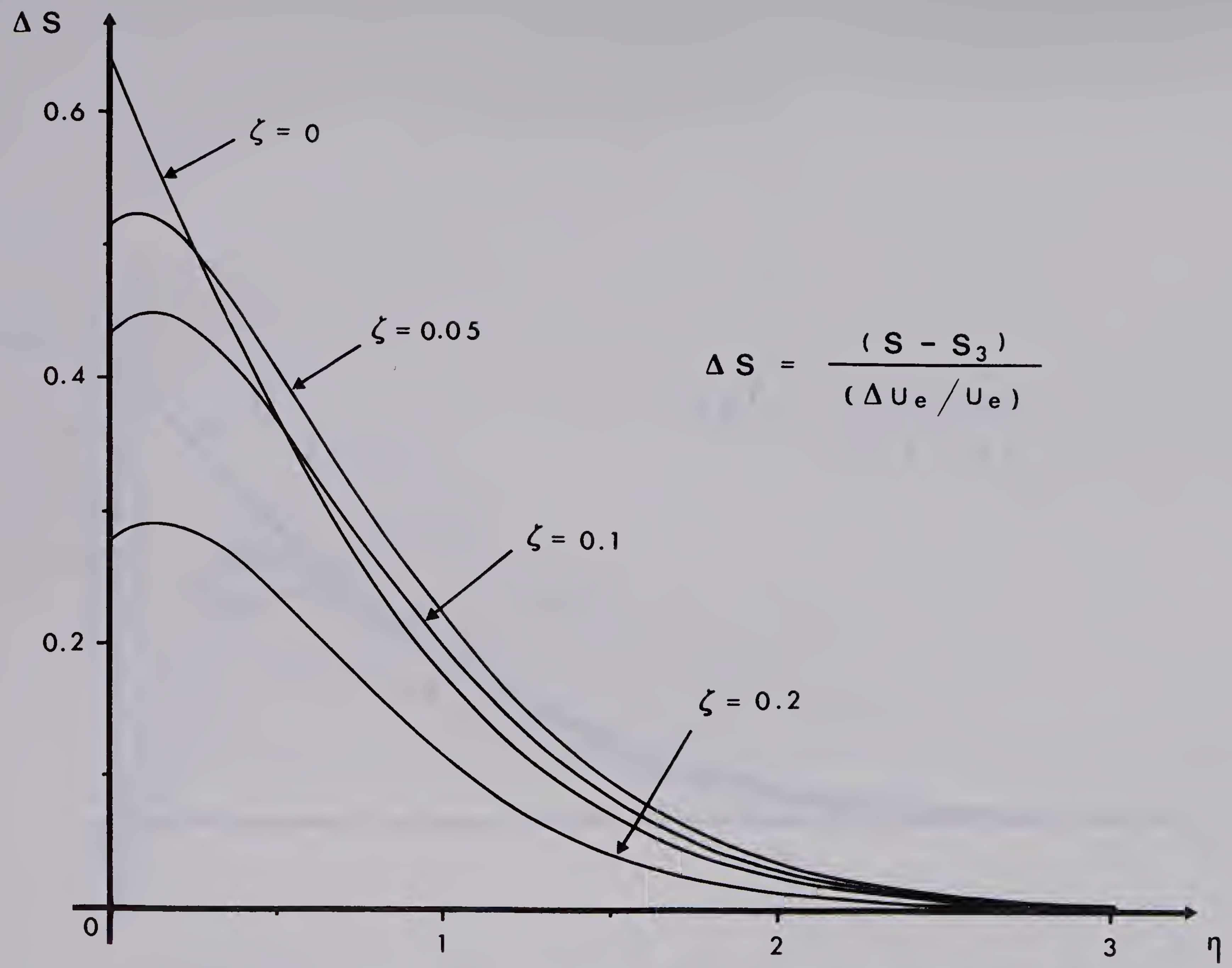

FIGURE 7c. Transient Contribution to the Total Enthalpy Function at $\xi = 0.3$ with $\theta_p / \theta_o = 1$

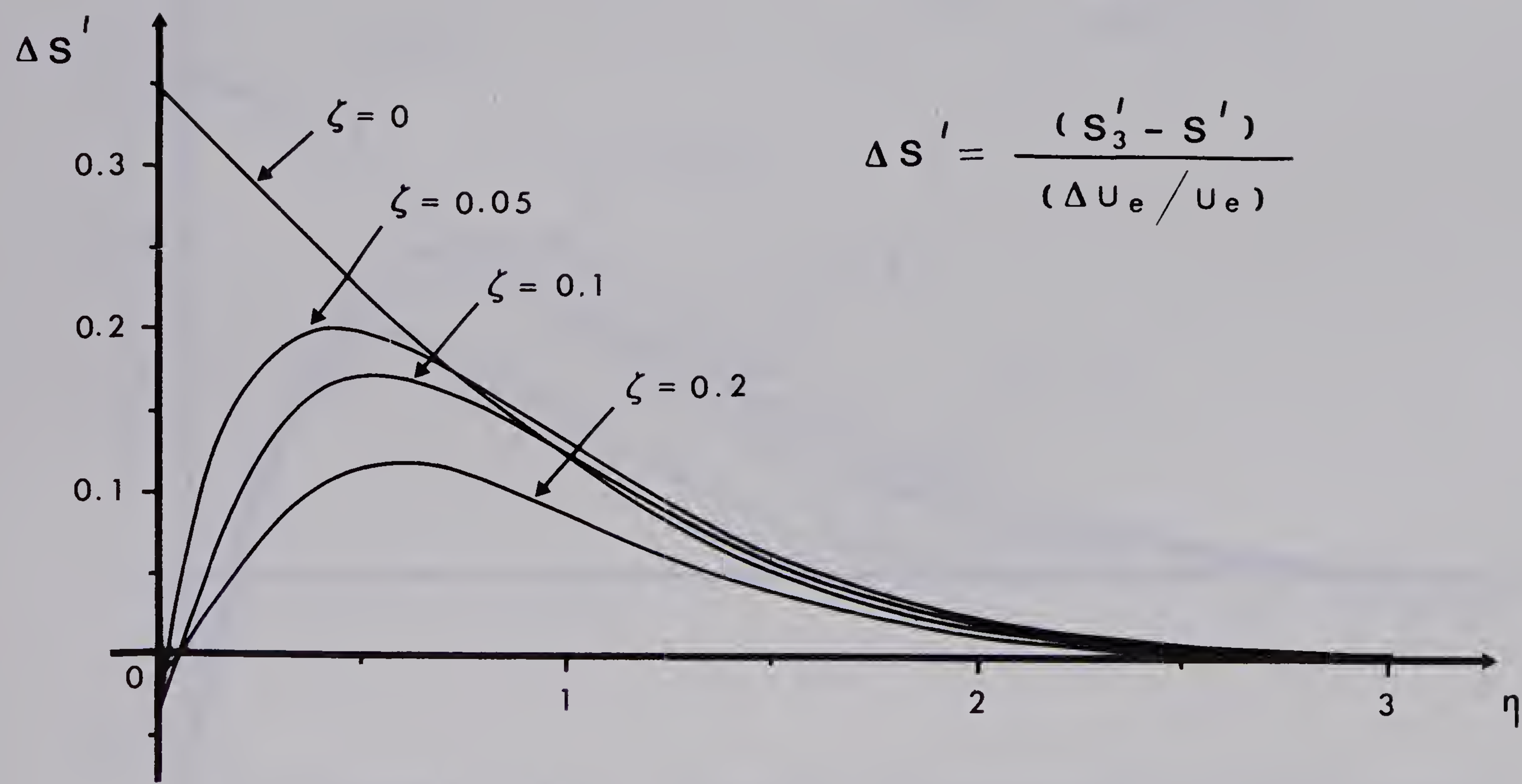

FIGURE 8a. Transient Contribution to the Total Enthalpy Gradient at $\xi = 0.1$ with $\theta_p / \theta_o = 0$

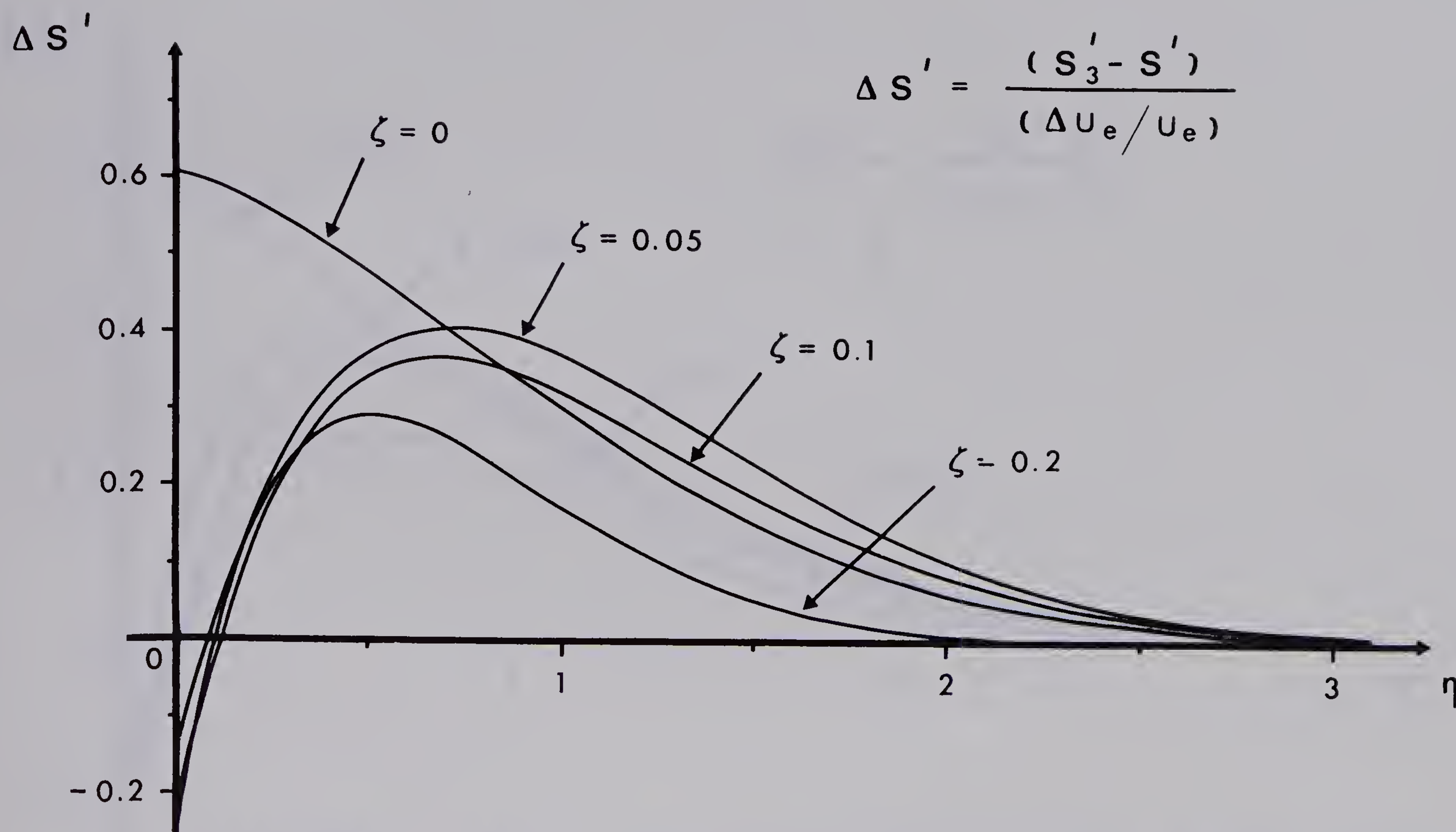

FIGURE 8b. Transient Contribution to the Total Enthalpy Gradient at $\xi = 0.3$ with $\theta_p / \theta_o = 0$

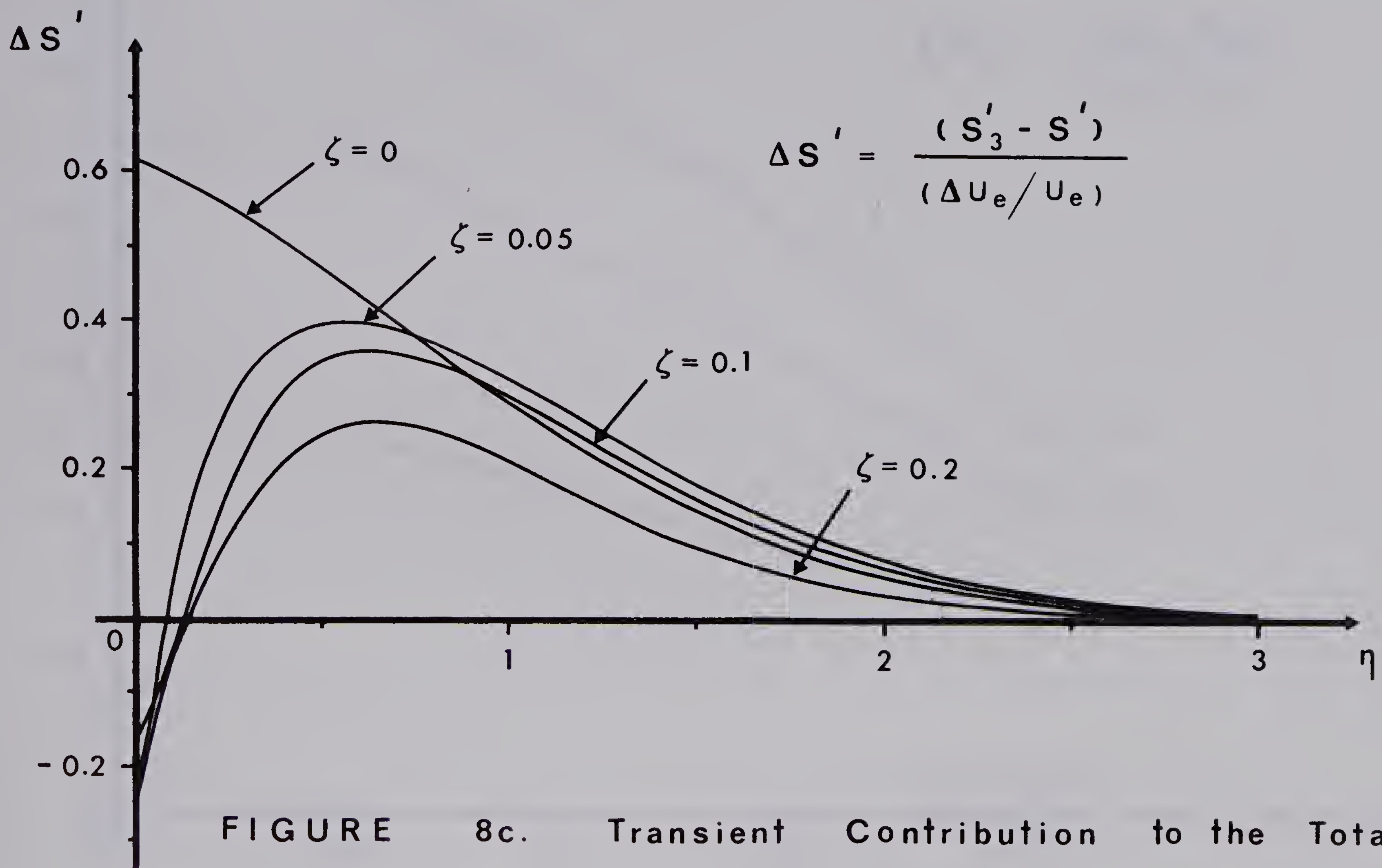

FIGURE 8c. Transient Contribution to the Total Enthalpy Gradient at $\xi = 0.3$ with $\theta_p / \theta_o = 1$

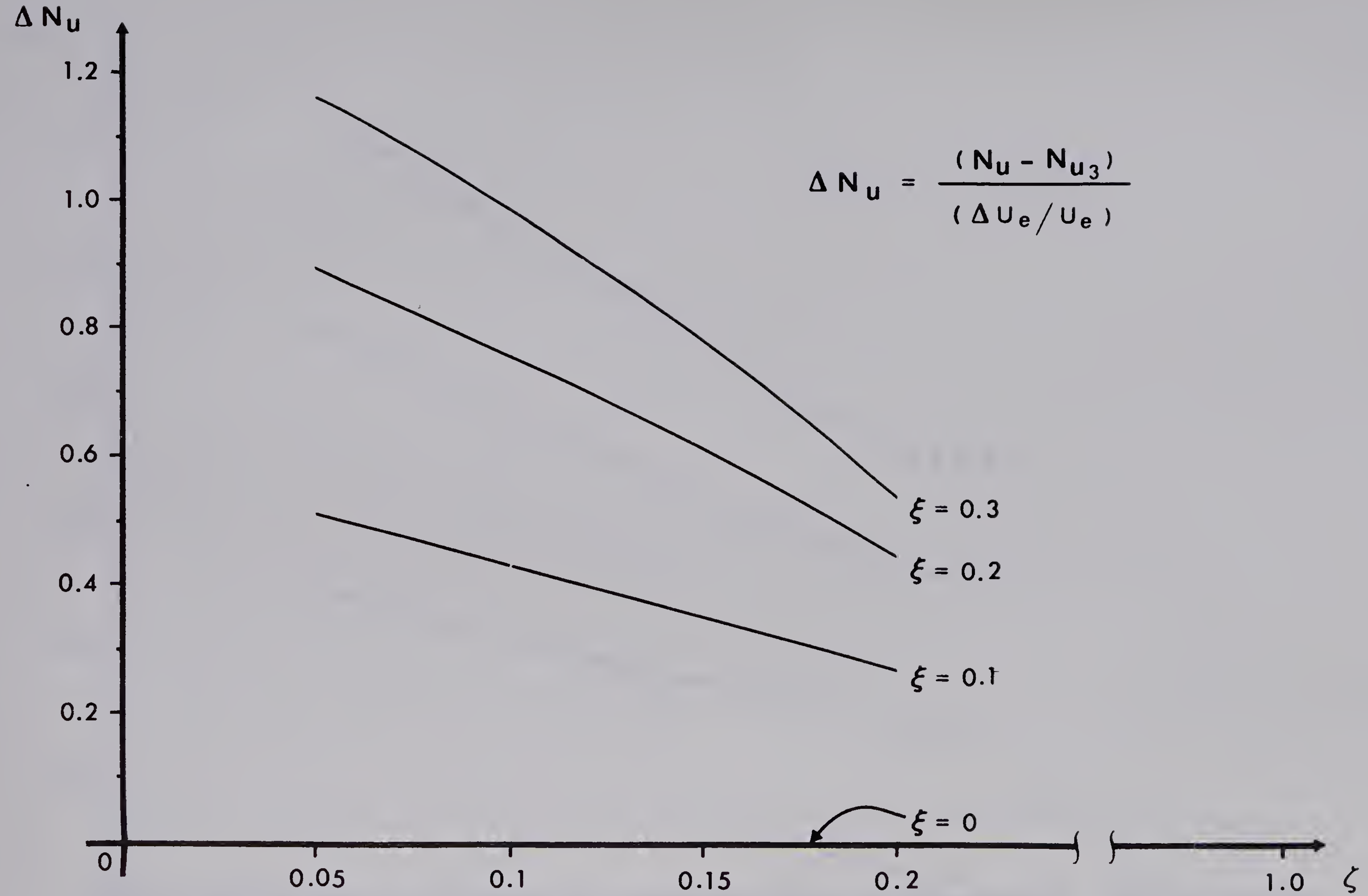

FIGURE 9a. Transient Contribution to the Heat Transfer Function with $\theta_p / \theta_o = 0$

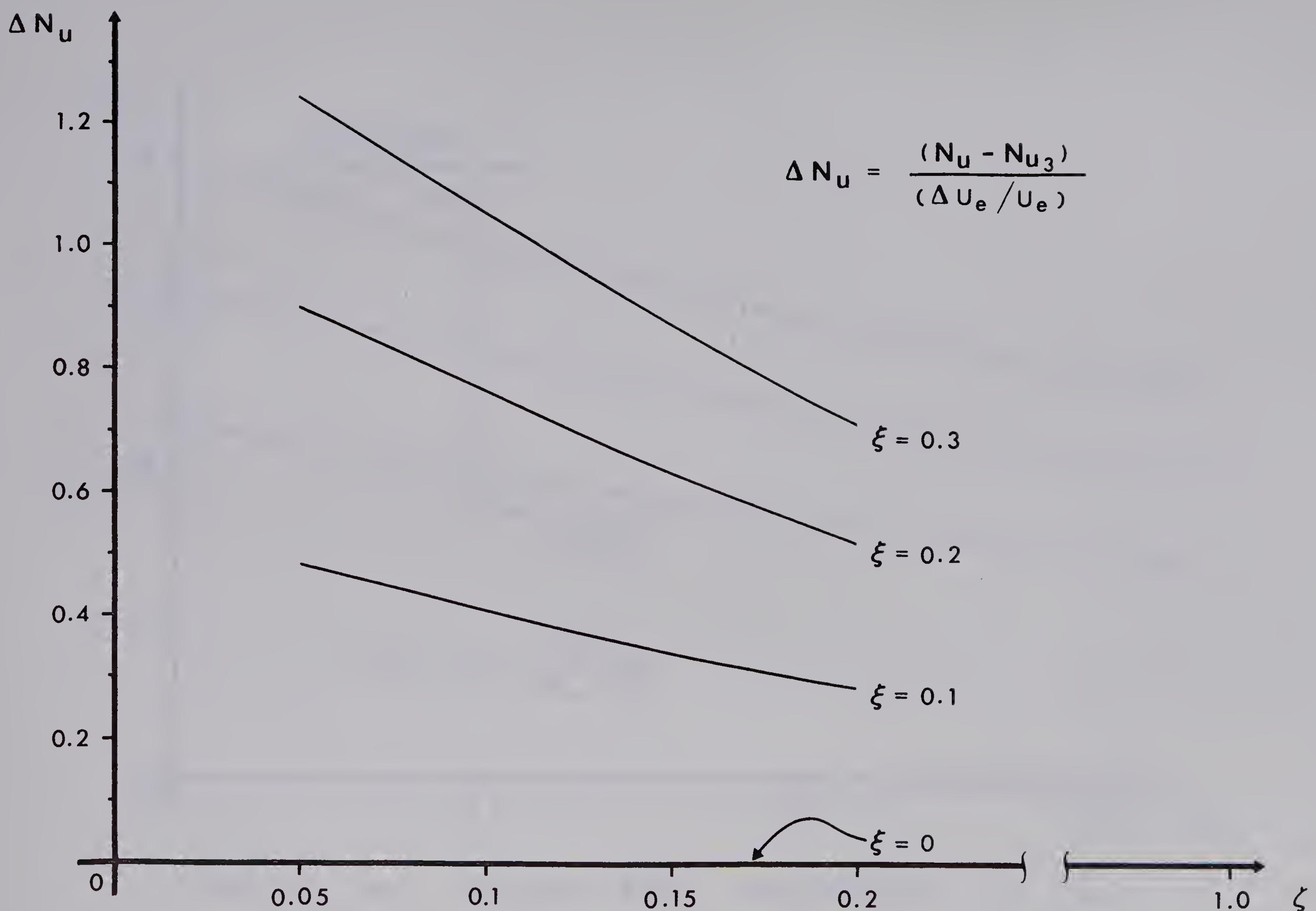

FIGURE 9b. Transient Contribution to the Heat Transfer Function with $\theta_p/\theta_o = 1$

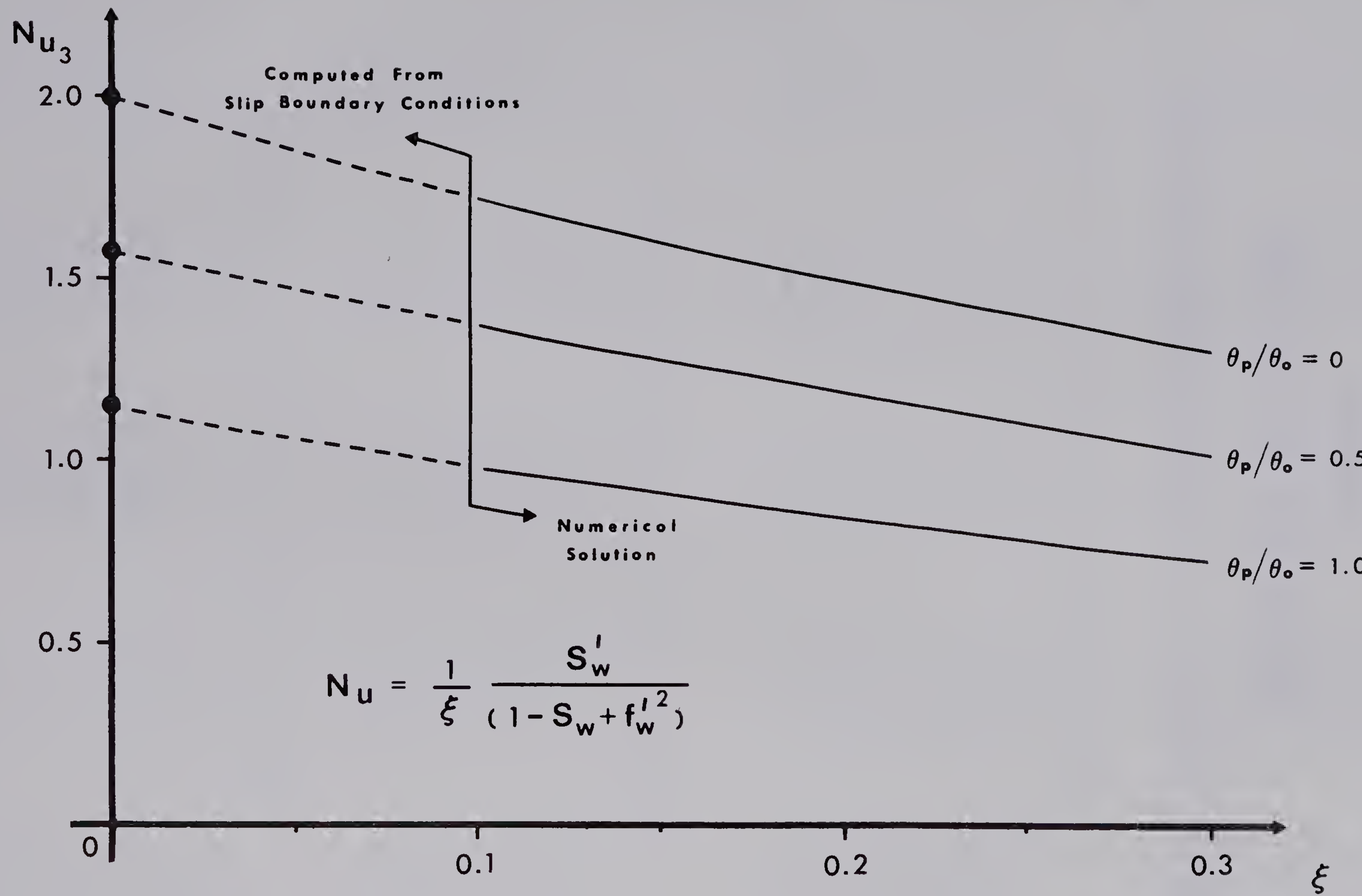

FIGURE 9c. Steady - State Distribution of the Heat Transfer Function

APPENDIX E

TABLES OF STEADY STATE SOLUTIONS

- the steady-state profiles of f', f'', S, and S' in tabular form.

$\xi = 0.1$, $\Delta\eta = 0.1$

STEADY STATE (WITH SLIP) VELOCITY DISTRIBUTION:

f'

0.92364	0.93253	0.94071	0.94820	0.95501
0.96117	0.96669	0.97161	0.97596	0.97979
0.98312	0.98600	0.98847	0.99058	0.99235
0.99384	0.99508	0.99610	0.99693	0.99760
0.99814	0.99857	0.99891	0.99918	0.99938
0.99954	0.99966	0.99975	0.99982	0.99987
0.99991	0.99993	0.99995	0.99997	0.99998
0.99998	0.99999	0.99999	0.99999	0.99999
0.99999	1.00000			

STEADY STATE (WITH SLIP) SHEAR DISTRIBUTION:

f''

0.09236	0.08532	0.07834	0.07148	0.06479
0.05834	0.05216	0.04631	0.04082	0.03571
0.03100	0.02670	0.02282	0.01935	0.01627
0.01357	0.01122	0.00920	0.00748	0.00603
0.00482	0.00382	0.00300	0.00233	0.00180
0.00137	0.00104	0.00078	0.00058	0.00043
0.00031	0.00023	0.00016	0.00011	0.00008
0.00006	0.00004	0.00003	0.00002	0.00001
0.00001	0.00000			

$\xi = 0.2$, $\Delta\eta = 0.1$

STEADY STATE (WITH SLIP) VELOCITY DISTRIBUTION:

f'

0.85260	0.86907	0.88435	0.89845	0.91136
0.92310	0.93372	0.94324	0.95171	0.95920
0.96577	0.97148	0.97642	0.98064	0.98422
0.98724	0.98976	0.99185	0.99356	0.99495
0.99608	0.99697	0.99768	0.99824	0.99868
0.99901	0.99927	0.99946	0.99961	0.99972
0.99980	0.99986	0.99990	0.99993	0.99995
0.99996	0.99997	0.99998	0.99999	0.99999
0.99999	0.99999	0.99999	1.00000	

STEADY STATE (WITH SLIP) SHEAR DISTRIBUTION:

f''

0.17052	0.15876	0.14688	0.13500	0.12325
0.11174	0.10059	0.08989	0.07973	0.07018
0.06129	0.05311	0.04565	0.03891	0.03290
0.02758	0.02293	0.01889	0.01544	0.01250
0.01004	0.00799	0.00630	0.00492	0.00381
0.00293	0.00223	0.00168	0.00125	0.00093
0.00068	0.00049	0.00035	0.00025	0.00018
0.00012	0.00009	0.00006	0.00004	0.00003
0.00002	0.00001	0.00001	0.00000	

$\xi = 0.5$, $\Delta\eta = 0.1$

STEADY STATE (WITH SLIP) VELOCITY DISTRIBUTION:

	0.78719	0.81006	0.83141	0.85122	0.86949
	0.88622	0.90142	0.91515	0.92746	0.93840
	0.94805	0.95651	0.96385	0.97017	0.97557
	0.98015	0.98399	0.98719	0.98983	0.99199
f'	0.99375	0.99515	0.99628	0.99716	0.99786
	0.99839	0.99881	0.99912	0.99936	0.99953
	0.99966	0.99976	0.99983	0.99988	0.99992
	0.99994	0.99996	0.99997	0.99998	0.99998
	0.99999	0.99999	0.99999	0.99999	0.99999
	1.00000				

STEADY STATE (WITH SLIP) SHEAR DISTRIBUTION:

	0.23616	0.22117	0.20586	0.19040	0.17493
	0.19561	0.14461	0.13006	0.11610	0.10285
	0.09040	0.07882	0.06817	0.05847	0.04973
	0.04194	0.03507	0.02907	0.02388	0.01945
f''	0.01570	0.01256	0.00996	0.00783	0.00609
	0.00470	0.00359	0.00272	0.00204	0.00152
	0.00112	0.00082	0.00059	0.00042	0.00030
	0.00021	0.00015	0.00010	0.00007	0.00005
	0.00003	0.00002	0.00001	0.00001	0.00001
	0.00000				

$\xi = 0.1$, $\Delta\eta = 0.1$, $\theta_p/\theta_o = 1$

STEADY STATE (WITH SLIP) ENTHALPY DISTRIBUTION:

	0.92473	0.93349	0.94156	0.94894	0.95565
	0.96172	0.96716	0.97201	0.97631	0.98007
	0.98336	0.98620	0.98864	0.98071	0.99246
	0.99393	0.99515	0.99616	0.99698	0.99764
S	0.99817	0.99860	0.99893	0.99919	0.99940
	0.99955	0.99967	0.99976	0.99983	0.99988
	0.99991	0.99994	0.99996	0.99997	0.99998
	0.99999	0.99999	0.99999	1.00000	1.00000
	1.00000	1.00000			

STEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION:

	0.09105	0.08411	0.07722	0.07046	0.06387
	0.05751	0.05142	0.04565	0.04024	0.03520
	0.03056	0.02632	0.02250	0.01907	0.01604
	0.01338	0.01106	0.00907	0.00738	0.00595
S'	0.00475	0.00376	0.00295	0.00230	0.00177
	0.00136	0.00103	0.00077	0.00057	0.00042
	0.00031	0.00022	0.00016	0.00011	0.00008
	0.00006	0.00004	0.00003	0.00002	0.00001
	0.00001	0.00001			

$\xi = 0.2$, $\Delta\eta = 0.1$, $\theta_p/\theta_o = 1$

STEADY STATE (WITH SLIP) ENTHALPY DISTRIBUTION:

S

0.86861	0.88257	0.89568	0.90790	0.91921
0.92959	0.93904	0.94759	0.95525	0.96206
0.96806	0.97331	0.97787	0.98178	0.98512
0.98794	0.99030	0.99226	0.99388	0.99519
0.99626	0.99711	0.99779	0.99832	0.99874
0.99906	0.99930	0.99949	0.99963	0.99973
0.99981	0.99986	0.99991	0.99993	0.99996
0.99997	0.99998	0.99999	0.99999	0.99999
1.00000	1.00000	1.00000	1.00000	

STEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION:

S′

0.14363	0.13541	0.12671	0.11767	0.10844
0.09916	0.08996	0.08097	0.07228	0.06400
0.05619	0.04893	0.04224	0.03616	0.03068
0.02581	0.02153	0.01779	0.01458	0.01183
0.00952	0.00759	0.00600	0.00470	0.00364
0.00280	0.00213	0.00161	0.00120	0.00089
0.00065	0.00048	0.00034	0.00024	0.00017
0.00012	0.00008	0.00006	0.00004	0.00003
0.00002	0.00001	0.00001	0.00001	

$\xi = 0.3$, $\Delta\eta = 0.1$, $\theta_p/\theta_o = 1$

STEADY STATE (WITH SLIP) ENTHALPY DISTRIBUTION:

S

0.83054	0.84707	0.86283	0.87773	0.89172
0.90475	0.91677	0.92779	0.93779	0.94679
0.95483	0.96194	0.96818	0.97360	0.97827
0.98225	0.98562	0.98845	0.99079	0.99272
0.99430	0.99557	0.99658	0.99739	0.99802
0.99852	0.99889	0.99918	0.99940	0.99957
0.99969	0.99978	0.99984	0.99989	0.99993
0.99995	0.99997	0.99998	0.99999	0.99999
0.99999	1.00000	1.00000	1.00000	1.00000

STEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION:

S′

0.16888	0.16157	0.15341	0.14455	0.13513
0.12529	0.11522	0.10507	0.09500	0.08516
0.07567	0.06664	0.05818	0.05033	0.04315
0.03666	0.03086	0.02574	0.02128	0.01742
0.01413	0.01136	0.00905	0.00714	0.00558
0.00432	0.00331	0.00252	0.00189	0.00141
0.00104	0.00076	0.00055	0.00040	0.00028
0.00020	0.00014	0.00010	0.00007	0.00004
0.00003	0.00002	0.00001	0.00001	0.00001

$\xi = 0.1$, $\Delta\eta = 0.1$, $\theta_p/\theta_o = 0.5$

STEADY STATE (WITH SLIP) ENTHALPY DISTRIBUTION:

S

0.89165	0.90426	0.91587	0.92649	0.93616
0.94489	0.95273	0.95971	0.96589	0.97132
0.97604	0.98013	0.98364	0.98663	0.98915
0.99127	0.99302	0.99447	0.99565	0.99660
0.99737	0.99798	0.99846	0.99884	0.99913
0.99935	0.99952	0.99965	0.99975	0.99982
0.99987	0.99991	0.99994	0.99996	0.99997
0.99998	0.99999	0.99999	1.00000	1.00000
1.00000	1.00000			

STEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION:

S'

0.13107	0.12108	0.11117	0.10143	0.09194
0.08278	0.07402	0.06572	0.05792	0.05067
0.04399	0.03789	0.03239	0.02746	0.02309
0.01926	0.01592	0.01306	0.01062	0.00856
0.00684	0.00542	0.00425	0.00331	0.00255
0.00195	0.00148	0.00111	0.00083	0.00061
0.00044	0.00032	0.00023	0.00016	0.00012
0.00008	0.00006	0.00004	0.00003	0.00002
0.00001	0.00001			

$\xi = 0.2$, $\Delta\eta = 0.1$, $\theta_p/\theta_o = 0.5$

STEADY STATE (WITH SLIP) ENTHALPY DISTRIBUTION:

S

0.80429	0.82546	0.84525	0.86364	0.88058
0.89609	0.91018	0.92288	0.93423	0.94431
0.95317	0.96091	0.96761	0.97337	0.97826
0.98240	0.98585	0.98872	0.99108	0.99300
0.99455	0.99580	0.99678	0.99756	0.99816
0.99863	0.99899	0.99926	0.99946	0.99961
0.99972	0.99980	0.99986	0.99991	0.99994
0.99996	0.99997	0.99998	0.99999	0.99999
1.00000	1.00000	1.00000	1.00000	

STEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION:

S'

0.21832	0.20489	0.19094	0.17667	0.16228
0.14794	0.13385	0.12017	0.10703	0.09457
0.08289	0.07205	0.06211	0.05309	0.04500
0.03781	0.03150	0.02601	0.02129	0.01727
0.01388	0.01106	0.00874	0.00684	0.00530
0.00407	0.00310	0.00234	0.00175	0.00129
0.00095	0.00069	0.00050	0.00035	0.00025
0.00018	0.00012	0.00008	0.00006	0.00004
0.00003	0.00002	0.00001	0.00001	

$\xi = 0.3$, $\Delta\eta = 0.1$, $\theta_p/\theta_o = 0.5$

STEADY STATE (WITH SLIP) ENTHALPY DISTRIBUTION:

	0.73697	0.76361	0.78880	0.81245	0.83449
	0.85488	0.87359	0.89064	0.90605	0.91985
	0.93212	0.94294	0.95239	0.96058	0.96761
	0.97359	0.97864	0.98286	0.98636	0.98924
S	0.99157	0.99346	0.99496	0.99616	0.99709
	0.99782	0.99838	0.99880	0.99913	0.99937
	0.99955	0.99968	0.99977	0.99984	0.99989
	0.99993	0.99995	0.99997	0.99998	0.99999
	0.99999	1.00000	1.00000	1.00000	1.00000

STEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION:

	0.27339	0.25932	0.24430	0.22852	0.21219
	0.19554	0.17881	0.16221	0.14597	0.13027
	0.11529	0.10117	0.08802	0.07591	0.06490
	0.05499	0.04618	0.03844	0.03170	0.02591
S'	0.02098	0.01684	0.01339	0.01055	0.00823
	0.00637	0.00488	0.00370	0.00278	0.00207
	0.00153	0.00112	0.00081	0.00058	0.00041
	0.00029	0.00020	0.00014	0.00010	0.00006
	0.00004	0.00003	0.00002	0.00001	0.00001

$\xi = 0.1$, $\Delta\eta = 0.1$, $\theta_p/\theta_o = 0$

STEADY STATE (WITH SLIP) ENTHALPY DISTRIBUTION:

	0.85856	0.87502	0.89018	0.90405	0.91667
	0.92807	0.93830	0.94741	0.95548	0.96256
	0.96873	0.97407	0.97865	0.98255	0.98584
	0.98860	0.99089	0.99278	0.99432	0.99556
S	0.99657	0.99736	0.99799	0.99848	0.99885
	0.99916	0.99938	0.99955	0.99967	0.99977
	0.99983	0.99988	0.99992	0.99994	0.99996
	0.99998	0.99998	0.99999	0.99999	1.00000
	1.00000	1.00000			

STEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION:

	0.17109	0.15805	0.14511	0.13240	0.12002
	0.10806	0.09662	0.08579	0.07561	0.06614
	0.05742	0.04946	0.04227	0.03584	0.03014
	0.02513	0.02079	0.01705	0.01386	0.01117
S'	0.00893	0.00707	0.00555	0.00432	0.00333
	0.00255	0.00193	0.00145	0.00108	0.00079
	0.00058	0.00042	0.00030	0.00021	0.00015
	0.00011	0.00007	0.00005	0.00003	0.00002
	0.00002	0.00001			

$\xi = 0.2$, $\Delta\eta = 0.1$, $\theta_p/\theta_o = 0$

STEADY STATE (WITH SLIP) ENTHALPY DISTRIBUTION:

	0.73997	0.76835	0.79483	0.81937	0.84195
	0.86260	0.88132	0.89817	0.91322	0.92656
	0.93828	0.94851	0.95736	0.96495	0.97141
	0.97685	0.98141	0.98518	0.98828	0.99081
S	0.99285	0.99449	0.99578	0.99680	0.99759
	0.99820	0.99867	0.99903	0.99929	0.99949
	0.99964	0.99974	0.99982	0.99988	0.99992
	0.99994	0.99996	0.99998	0.99998	0.99999
	0.99999	1.00000	1.00000	1.00000	

STEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION:

	0.29301	0.27437	0.25518	0.23567	0.21611
	0.19672	0.17774	0.15937	0.14179	0.12515
	0.10959	0.09518	0.08198	0.07003	0.05931
	0.04981	0.04146	0.03422	0.02800	0.02270
S'	0.01825	0.01453	0.01147	0.00898	0.00696
	0.00534	0.00407	0.00307	0.00229	0.00170
	0.00124	0.00090	0.00065	0.00046	0.00033
	0.00023	0.00016	0.00011	0.00007	0.00005
	0.00003	0.00002	0.00001	0.00001	

$\xi = 0.3$, $\Delta\eta = 0.1$, $\theta_p/\theta_o = 0$

STEADY STATE (WITH SLIP) ENTHALPY DISTRIBUTION:

	0.64339	0.68015	0.71477	0.74716	0.77725
	0.80501	0.83041	0.85349	0.87430	0.89291
	0.90941	0.92393	0.93660	0.94755	0.95695
	0.96493	0.97166	0.97728	0.98193	0.98575
S	0.98885	0.99135	0.99335	0.99492	0.99615
	0.99712	0.99786	0.99842	0.99885	0.99917
	0.99940	0.99957	0.99970	0.99979	0.99986
	0.99990	0.99993	0.99996	0.99997	0.99998
	0.99999	0.99999	1.00000	1.00000	1.00000

STEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION:

	0.37790	0.35708	0.33518	0.31248	0.28925
	0.26579	0.24239	0.21935	0.19693	0.17538
	0.15491	0.13569	0.11786	0.10149	0.08664
	0.07332	0.06150	0.05113	0.04213	0.03440
S'	0.02783	0.02232	0.01773	0.01396	0.01089
	0.00841	0.00644	0.00489	0.00367	0.00273
	0.00202	0.00147	0.00107	0.00076	0.00054
	0.00038	0.00027	0.00018	0.00013	0.00008
	0.00006	0.00004	0.00002	0.00002	0.00001

APPENDIX F

THE COMPUTER PROGRAMS

- written in FORTRAN IV for an IBM 360/67 computer using the MTS system. Data generated by the programs is stored on magnetic disks using the MTS FILE system.

F.1 The Momentum Equation Program

```
C     MAIN PROGRAM
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION F(3,71,3),FN(3,71,3),FNN(3,71,3),FNNN(71)
      DIMENSION PRMT(5),FUNC(3),DERF(3),AUX(16,3)
      DIMENSION STORE(3,71,2,3),STOREA(26,71,3),STOREB(3,71,3)
      DIMENSION FNNS(3),ISOL(2)
C
      COMMON F,FN,FNN,FNNN,ISOL,NE/FCTDAT/EQN,X,DX,T,DT,G
      COMMON /INTDAT/IX,MPS
C
      EXTERNAL FCT,OUTP
C
      IO=0
      IO=1
C
      KT=2
      KT=3
      KT=1
      KT=0
C
      NS=2
      NS=3
C
      UE3=1.01D0
C
      IEX=5
      IEX=3
C
      IBSR=0
      NDIM=3
C
      PRMT(2)=7D0
      PRMT(3)=0.1D0
      PRMT(4)=5D-2
C
      ITERS=30
      ITERS=40
      ITERS=50
      IF (KT.GT.0) ITERS=30
C
      DO 6 L=1,3
C
      DO 6 J=1,71
C
      DO 6 I=1,26
C
      STOREA(I,J,L)=0D0
      IF (I.GT.3) GO TO 5
      STORE(I,J,1,L)=0D0
      STORE(I,J,2,L)=0D0
      STOREB(I,J,L)=0D0
C
    5 CONTINUE
C
```

```
    6 CONTINUE
C
      DO 10 J=1,71
C
      FNNN(J)=0D0
C
      DO 10 I=1,3
C
      DO 10 K=1,3
C
      FNN(I,J,K)=0D0
      FN(I,J,K)=0D0
      F(I,J,K)=0D0
C
   10 CONTINUE
C
      IF (KT.NE.1) GO TO 12
C
      READ (1) STOREA
   15 CONTINUE
C
      DO 11 I=1,3
      DO 11 L=1,3
      DO 11 J=1,71
C
      STORE(I,J,1,L)=STOREA(I,J,L)
C
   11 CONTINUE
      IF (NS.EQ.1) GO TO 22
C
   12 CONTINUE
C
      IF (KT.LT.2.OR.KT.GT.3) GO TO 14
C
      READ (1) STOREA
      IF (KT.EQ.2) READ (2) STOREB
      IF (KT.EQ.3) READ (3) STOREB
C
      DO 13 I=1,3
      DO 13 L=1,3
      DO 13 J=1,71
C
      STORE(I,J,2,L)=STOREA(I,J,L)
      STORE(I,J,1,L)=STOREB(I,J,L)
C
   13 CONTINUE
C
   14 CONTINUE
C
      DO 20 I=1,3
C
      FNNS(I)=0D0
C
   20 CONTINUE
C
```

```
      G=5D0/3D0
      G=1.4D0
      TMACCO=0.6D0
      TMACCO=1D0
C
      DX=0.1D0
      DN=0.1D0
      DT=0.0125D0
      DT=0.05D0
C
      FNNW=0.1D0
C
      DFNNW=0.1D0
C
      PI=3.14159265358979D0
      DEX=DX
      DTEE=DT
      X=0D0
      T=0D0
      C1=(2D0-TMACCO)/TMACCO
      NE=71
C
      DT=8D0*DTEE
      IF (KT.GE.12) DT=0.01D0
      IF (KT.LE.9) DT=4D0*DTEE
      IF (KT.LE.5) DT=2D0*DTEE
      IF (KT.LE.2) DT=DTEE
C
      IF (KT.GE.12) T=0.5D0+DFLOAT(KT-11)*DT
      IF (KT.LE.11) T=0.3D0+DFLOAT(KT-9)*DT
      IF (KT.LE.9) T=0.1D0+DFLOAT(KT-5)*DT
      IF (KT.LE.5) T=0.1D0+DFLOAT(KT-2)*DT
      IF (KT.LE.2) T=DFLOAT(KT)*DT
C
   22 CONTINUE
      DO 180 IX=1,IEX
C
      PRMT(1)=0D0
      IF (IX.EQ.26) PRMT(4)=5D-5
      IF (KT.EQ.0.AND.NS.NE.1) GO TO 26
C
      DO 23 K=2,3
C
      KKK=K-1
      L=K
      IF (NS.EQ.1) L=1
      IF (NS.EQ.1) KKK=1
C
      DO 23 J=1,71
C
      F(1,J,L)=STORE(IX,J,KKK,1)
      FN(1,J,L)=STORE(IX,J,KKK,2)
      FNN(1,J,L)=STORE(IX,J,KKK,3)
C
   23 CONTINUE
```

```
C
      IF (NS.EQ.1) GO TO 98
   26 CONTINUE
C
      IF (IX.EQ.1) GO TO 35
C
      IJ=1
C
      IF (IX.EQ.2.OR.IX.EQ.7.OR.IX.EQ.14.OR.IX.EQ.20) IJ=2
C
      DO 30 II=IJ,2
C
      I=4-II
      IJI=I-1
C
      DO 30 J=1,71
C
      F(I,J,1)=F(IJI,J,1)
      FN(I,J,1)=FN(IJI,J,1)
      FNN(I,J,1)=FNN(IJI,J,1)
C
   30 CONTINUE
C
   35 CONTINUE
C
      MDPTCR=0
      LF=0
      LF1=0
C
      DX=8D0*DEX
      IF (IX.LE.19) DX=4D0*DEX
      IF (IX.LE.13) DX=2D0*DEX
      IF (IX.LE.6) DX=DEX
      X=X+DX
C
      EQN=2D0
      IF (IX.GT.1.AND.KT.EQ.0) EQN=3D0
      IF (IX.EQ.1.AND.KT.EQ.1.OR.KT.EQ.12) EQN=22D0
      IF (IX.GT.1.AND.KT.EQ.1.OR.KT.EQ.12) EQN=32D0
      IF (IX.EQ.1.AND.KT.GT.1) EQN=23D0
      IF (IX.GT.1.AND.KT.GT.1) EQN=33D0
C
   37 CONTINUE
C
      IBSR=0
      ISOL(1)=0
C
      MPS=1
      IF (MDPTCR.EQ.1) MPS=11
C
      IF (KT.EQ.0.AND.IX.GT.2) FNNW=2D0*FNN(2,1,1)-FNN(3,1,1)
      IF (KT.EQ.1) FNNW=0.99D0*FNN(1,1,2)
      IF (KT.GT.1) FNNW=2D0*FNN(1,1,2)-FNN(1,1,3)
      IF (MPS.EQ.11) FNNW=FNN(1,11,1)
C
```

```
      IF (KT.EQ.0.AND.IX.GT.2) DFNNW=DABS(FNN(2,1,1)-FNN(3,1,1))/10D0
      IF (KT.EQ.1) DFNNW=0.01D0*FNN(1,1,2)
      IF (KT.GT.1) DFNNW=DABS(FNN(1,1,2)-FNN(1,1,3))/10D0
      IF (MPS.EQ.11) DFNNW=0.01D0*FNNW
C
      IF (MPS.EQ.11) FNNA=FNNW
C
   40 CONTINUE
C
      IF (MDPTCR.EQ.0) LF=LF+1
      IF (MDPTCR.EQ.1) LF1=LF1+1
      ISOL(2)=0
C
      DERF(1)=0.1D0
      DERF(2)=0.3D0
      DERF(3)=0.6D0
C
      IF (MDPTCR.EQ.1) GO TO 63
C
      FUNC(3)=FNNW/0.9D0
C
   60 CONTINUE
C
      FUNC(3)=0.9D0*FUNC(3)
      FUNC(2)=C1*FUNC(3)/X-1D0
      IF (FUNC(2).GT.0D0) GO TO 60
      FUNC(1)=0D0
C
      GO TO 66
C
   63 CONTINUE
C
      PRMT(1)=1D0
      FUNC(3)=FNNW
      FUNC(2)=FN(1,11,1)
      FUNC(1)=F(1,11,1)
C
   66 CONTINUE
C
      FNNS(2)=FUNC(3)
C
      CALL DHPCG(PRMT,FUNC,DERF,NDIM,IHLF,FCT,OUTP,AUX)
C
      IF (IHLF.GE.11) WRITE (6,70) IHLF
   70 FORMAT (' '/T10,'IHLF=',I2/)
      IF (IHLF.GE.11) STOP
C
      ERROR=5D-5
      ERROR=5D-6
      IF (IX.GT.24) ERROR=5D-4
C
      IF (DABS(FUNC(2)).LT.ERROR.AND.DABS(FUNC(3)).LT.ERROR) GO TO 90
      IF (LF1.EQ.50) GO TO 90
      IF (LF.EQ.1.OR.(LF1.EQ.1.AND.MPS.EQ.11)) ISOL(1)=ISOL(2)
      IF ((LF.NE.1.OR.(LF1.NE.1.AND.MPS.EQ.11)).AND.(ISOL(1).NE.
```

```
     1ISOL(2).OR.IBSR.GT.0)) GO TO 80
C
      FNNS(ISOL(2))=FNNS(2)
      IF (ISOL(2).EQ.1) FNNW=FNNW+DFNNW
      IF (ISOL(2).EQ.3) FNNW=FNNW-DFNNW
C
      GO TO 40
C
   80 CONTINUE
C
      IBSR=1
C
      FNNS(ISOL(2))=FNNS(2)
      FNNW=(FNNS(1)+FNNS(3))/2D0
      IF (LF+LF1.EQ.ITERS) MDPTCR=1
      IF (LF+LF1.EQ.ITERS) GO TO 37
C
      GO TO 40
C
   90 CONTINUE
C
      IF (NE.EQ.71) GO TO 96
C
      JS=NE+1
      DO 95 J=JS,71
C
      DEL=1D0-DFLOAT(71-J)/DFLOAT(71-NE)
      F(1,J,1)=DEL*F(1,J,1)
      FN(1,J,1)=DEL*FN(1,J,1)
      FNN(1,J,1)=DEL*FNN(1,J,1)
      FNNN(J)=DEL*FNNN(J)
C
   95 CONTINUE
C
   96 CONTINUE
C
      IF (MPS.EQ.1) GO TO 98
C
      DELFNN=(FNN(1,11,1)-FNNA)/2D0
C
      DO 97 J=2,20
C
      ETA=DFLOAT(J-1)/10D0
      IF (J.LE.11) FNN(1,J,1)=FNN(1,J,1)+ETA*DELFNN
      IF (J.GT.11) FNN(1,J,1)=FNN(1,J,1)-(2D0-ETA)*DELFNN
C
   97 CONTINUE
C
   98 CONTINUE
      IF (NS.EQ.2) GO TO 165
C
      DO 100 J=1,NE
C
      FN(1,J,1)=FN(1,J,1)+1D0
C
```

```
 100   CONTINUE
 C
       FSETA=DN*DFLOAT(NE-1)
 C
       IF (IX.EQ.1) WRITE (6,110)
 110   FORMAT ('1')
 C
       TS=T
       IF (KT.EQ.0.AND.NS.EQ.3) T=10D0
       WRITE (6,120) X,FSETA,T
       T=TS
 120   FORMAT ('6',T10,'X=',F4.1,T20,'ETA(FREE STREAM)=',F3.1,
      1T45,'T=',F6.4)
 C
       WRITE (6,130)
 130   FORMAT (' '///T10,'UNSTEADY STATE (WITH SLIP) VELOCITY DISTRIBUTI
      1N:'/)
 C
       WRITE (6,140) (FN(1,J,1),J=1,NE)
 140   FORMAT (' ',T2,5F23.5)
 C
       WRITE (6,150)
 150   FORMAT (' '///T10,'UNSTEADY STATE (WITH SLIP) SHEAR DISTRIBUTION:'
      1/)
 C
       WRITE (6,140) (FNN(1,J,1),J=1,NE)
 C
       LFS=LF+LF1
 C
       WRITE (6,155) LF,LF1,LFS
 155   FORMAT (' '/////T10,'CONVERGED IN ',I2,'+',I2,'=',I2,' ITERATIONS.
      1')
 C
       WRITE (6,110)
 C
       DO 160 J=1,NE
 C
       FN(1,J,1)=FN(1,J,1)-1D0
 C
 160   CONTINUE
 165   CONTINUE
 C
       IF (KT.EQ.0) GO TO 171
 C
       JI=2
       IF (KT.EQ.0.OR.KT.EQ.2.OR.KT.EQ.5.OR.KT.EQ.9.OR.KT.EQ.11) JI=1
 C
       DO 170 K=1,JI
 C
       DO 170 J=1,71
 C
       STORE(IX,J,K,1)=F(1,J,K)
       STORE(IX,J,K,2)=FN(1,J,K)
       STORE(IX,J,K,3)=FNN(1,J,K)
 C
```

```
 170   CONTINUE
C
 171   CONTINUE
C
       IF (KT.GT.0) GO TO 176
C
       DO 172 J=1,71
C
       STOREA(IX,J,1)=F(1,J,1)
       STOREA(IX,J,2)=FN(1,J,1)
       STOREA(IX,J,3)=FNN(1,J,1)
C
 172   CONTINUE
C
 176   CONTINUE
C
 180   CONTINUE
C
       IF (IO.EQ.0) GO TO 220
       IF (KT.NE.0) GO TO 200
       IF (NS.EQ.1) GO TO 220
       IF (NS.EQ.2) GO TO 190
C
       WRITE (5) STORFA
C
       GO TO 220
C
 190   CONTINUE
C
       CALL STATE2(STOREA)
C
       WRITE (1) STOREA
C
       NS=1
       GO TO 15
C
 200   CONTINUE
C
       DO 210 I=1,3
       DO 210 L=1,3
       DO 210 J=1,71
C
       STOREB(I,J,L)=STORE(1,J,1,L)
C
 210   CONTINUE
C
       IF (KT.EQ.1) WRITE (2) STOREB
       IF (KT.EQ.2) WRITE (3) STOREB
       IF (KT.EQ.3) WRITE (4) STOREB
C
 220   CONTINUE
C
       STOP
       END
```

```
      SUBROUTINE FCT(ETA,FUNC,DERF)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION F(3,71,3),FN(3,71,3),FNN(3,71,3),FNNN(71)
      DIMENSION FT(61,2),ARG(20),VAL(40)
      DIMENSION FUNC(3),DERF(3)
C
      COMMON F,FN,FNN,FNNN
      COMMON /FCTDAT/EQN,X,DX,T,DT,G
      COMMON /INTDAT/IX,MPS
C
      ERR=5D-5
      ERR=5D-6
      IF (IX.GT.24) ERR=5D-5
C
      N1=MPS+60
      N2=20
      N3=61
      EPS=ERR
C
      BGN=0D0
      IF (MPS.EQ.11) BGN=1D0
C
      DERF(1)=FUNC(2)
      DERF(2)=FUNC(3)
C
      INTERP=1
C
      DO 3 J=MPS,71
C
      N=J
C
      ETACHK=0.1D0*DFLOAT(N-1)
      DETA=ETA-ETACHK
C
      IF (DABS(DETA).LE.5D-10) INTERP=0
      IF (INTERP.EQ.0) ETA=ETACHK
      IF (INTERP.EQ.0) GO TO 6
      IF (DETA.LE.0D0) GO TO 6
C
    3 CONTINUE
C
    6 CONTINUE
C
      FM1X=F(2,N,1)
      FNM1X=FN(2,N,1)
C
      FM1T=F(1,N,2)
      FNM1T=FN(1,N,2)
C
      FM2X=F(3,N,1)
      FNM2X=FN(3,N,1)
C
      FM2T=F(1,N,3)
      FNM2T=FN(1,N,3)
```

```
C
      IF (INTERP.EQ.0) GO TO 50
C
      DO 10 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=F(2,J,1)
      FT(JJ,2)=FN(2,J,1)
C
   10 CONTINUE
C
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
C
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
C
      FM1X=P
C
      IF (IER.GT.1) WRITE (6,20) IER
   20 FORMAT (' '/T10,'IER=',I1,' FOR FM1X.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1) GO TO 10
      EPS=ERR
      IF (IER.GT.0) STOP
C
      DO 30 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=FN(2,J,1)
      FT(JJ,2)=FNN(2,J,1)
C
   30 CONTINUE
C
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
C
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
C
      FNM1X=P
C
      IF (IER.GT.1) WRITE (6,40) IER
   40 FORMAT (' '/T10,'IER=',I1,' FOR FNM1X.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1) GO TO 30
      EPS=ERR
      IF (IER.GT.0) STOP
C
   50 CONTINUE
C
      IF (EQN.NE.2D0) GO TO 60
C
      DERF(3)=-(FUNC(1)+ETA)*FUNC(3)-X*((FUNC(1)-FM1X)*FUNC(3)
     1-(FUNC(2)-FNM1X)*(FUNC(2)+1D0))/DX
C
      RETURN
C
   60 CONTINUE
```

```
C
      IF (INTERP.EQ.0.OR.EQN.EQ.3D0) GO TO 110
C
      DO 70 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=F(1,J,2)
      FT(JJ,2)=FN(1,J,2)
C
   70 CONTINUE
C
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
C
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
C
      FM1T=P
C
      IF (IER.GT.1) WRITE (6,80) IER
   80 FORMAT (' '/T10,'IER=',I1,' FOR FM1T.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1) GO TO 70
      EPS=ERR
      IF (IER.GT.0) STOP
C
      DO 90 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=FN(1,J,2)
      FT(JJ,2)=FNN(1,J,2)
C
   90 CONTINUE
C
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
C
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
C
      FNM1T=P
C
      IF (IER.GT.1) WRITE (6,100) IER
  100 FORMAT (' '/T10,'IER=',I1,' FOR FNM1T.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1) GO TO 90
      EPS=ERR
      IF ( IER.GT.0) STOP
C
  110 CONTINUE
C
      IF (EQN.NE.22D0) GO TO 120
C
      IF (MPS.EQ.11.AND.N.EQ.11) DERF(3)=FNNN(11)
      DERF(3)=-(FUNC(1)+ETA)*FUNC(3)-X*((FUNC(1)-FM1X)*FUNC(3)
     1-(FUNC(2)-FNM1X)*(FUNC(2)+1D0))/DX
     2+2D0*(1D0-T)**2*(FUNC(2)-FNM1T)/DT-2D0*T*(1D0-T)
     3*((FUNC(2)-FNM1T)*(FUNC(2)+1D0)-(FUNC(1)-FM1T)*FUNC(3))/DT
C
```

```
      RETURN
C
  120 CONTINUE
C
      IF (INTERP.EQ.0) GO TO 170
C
      DO 130 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=F(3,J,1)
      FT(JJ,2)=FN(3,J,1)
C
  130 CONTINUE
C
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
C
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
C
      FM2X=P
C
      IF (IER.GT.1) WRITE (6,140) IER
  140 FORMAT (' '/T10,'IER=',I1,' FOR FM2X.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1) GO TO 130
      EPS=ERR
      IF (IER.GT.0) STOP
C
      DO 150 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=FN(3,J,1)
      FT(JJ,2)=FNN(3,J,1)
C
  150 CONTINUE
C
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
C
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
C
      FNM2X=P
C
      IF (IER.GT.1) WRITE (6,160) IER
  160 FORMAT (' '/T10,'IER=',I1,' FOR FNM2X.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1) GO TO 150
      EPS=ERR
      IF (IER.GT.0) STOP
C
  170 CONTINUE
C
      IF (EQN.NE.300) GO TO 180
C
      DERF(3)=-(FUNC(1)+ETA)*FUNC(3)-X*((1.5D0*FUNC(1)-2D0*FM1X+0.5D0
     1*FM2X)*FUNC(3)-(1.5D0*FUNC(2)-2D0*FNM1X+0.5D0
     2*FNM2X)*(FUNC(2)+1D0))/DX
```

```
C
      RETURN
C
  180 CONTINUE
C
      IF (EQN.NE.32D0) GO TO 190
C
      DERF(3)=-(FUNC(1)+ETA)*FUNC(3)-X*((1.5D0*FUNC(1)-2D0*FM1X+0.5D0
     1*FM2X)*FUNC(3)-(1.5D0*FUNC(2)-2D0*FNM1X+0.5D0
     2*FNM2X)*(FUNC(2)+1D0))/DX
     3+2D0*(1D0-T)**2*(FUNC(2)-FNM1T)/DT-2D0*T*(1D0-T)
     4*((FUNC(2)-FNM1T)*(FUNC(2)+1D0)-(FUNC(1)-FM1T)*FUNC(3))/DT
C
      RETURN
C
  190 CONTINUE
C
      IF (INTERP.EQ.0) GO TO 240
C
      DO 200 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=F(1,J,3)
      FT(JJ,2)=FN(1,J,3)
C
  200 CONTINUE
C
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
C
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
C
      FM2T=P
C
      IF (IER.GT.1) WRITE (6,210) IER
  210 FORMAT (' '/T10,'IER=',I1,' FOR FM2T.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1) GO TO 200
      EPS=ERR
      IF (IER.GT.0) STOP
C
      DO 220 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=FN(1,J,3)
      FT(JJ,2)=FNN(1,J,3)
C
  220 CONTINUE
C
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
C
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
C
      FNM2T=P
C
      IF (IER.GT.1) WRITE (6,230) IER
```

```
 230  FORMAT (' '/T10,'IER=',I1,' FOR FNM2T.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1) GO TO 220
      EPS=ERR
      IF (IER.GT.0) STOP
C
 240  CONTINUE
C
      IF (EQN.NE.23D0) GO TO 250
C
      DERF(3)=-(FUNC(1)+ETA)*FUNC(3)-X*((FUNC(1)-FM1X)*FUNC(3)
     1-(FUNC(2)-FNM1X)*(FUNC(2)+1D0))/DX
     2+2D0*(1D0-T)**2*(1.5D0*FUNC(2)-2D0*FNM1T+0.5D0*FNM2T)/DT
     3-2D0*T*(1D0-T)*((1.5D0*FUNC(2)-2D0*FNM1T+0.5D0*FNM2T)
     4*(FUNC(2)+1D0)-(1.5D0*FUNC(1)-2D0*FM1T+0.5D0*FM2T)*FUNC(3))/DT
C
      RETURN
C
 250  CONTINUE
C
      DERF(3)=-(FUNC(1)+ETA)*FUNC(3)-X*((1.5D0*FUNC(1)-2D0*FM1X+0.5D0
     1*FM2X)*FUNC(3)-(1.5D0*FUNC(2)-2D0*FNM1X+0.5D0
     2*FNM2X)*(FUNC(2)+1D0))/DX
     3+2D0*(1D0-T)**2*(1.5D0*FUNC(2)-2D0*FNM1T+0.5D0*FNM2T)/DT
     4-2D0*T*(1D0-T)*((1.5D0*FUNC(2)-2D0*FNM1T+0.5D0*FNM2T)
     5*(FUNC(2)+1D0)-(1.5D0*FUNC(1)-2D0*FM1T+0.5D0*FM2T)*FUNC(3))/DT
C
      RETURN
      END
```

```
      SUBROUTINE OUTP(ETA,FUNC,DERF,IHLF,NDIM,PRMT)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION F(3,71,3),FN(3,71,3),FNN(3,71,3),FNNN(71)
      DIMENSION PRMT(5),FUNC(3),DERF(3),ISOL(2)
C
      COMMON F,FN,FNN,FNNN,ISOL,N
      COMMON /INTDAT/IX,MPS
C
      NX=3
      NX=4
      NX=1
      NX=2
C
      INTERP=1
C
      DO 3 J=1,71
C
      N=J
C
      ETACHK=0.1D0*DFLOAT(N-1)
      DETA=ETA-ETACHK
C
      IF (DABS(DETA).LE.5D-10) INTERP=0
      IF (INTERP.EQ.0) ETA=ETACHK
      IF (INTERP.EQ.0) GO TO 6
      IF (DETA.LE.0D0) GO TO 6
C
    3 CONTINUE
C
    6 CONTINUE
C
      IF ( IX.GE.NX) IO=1
      IO=1
      IO=0
C
      IF (INTERP.NE.0) RETURN
C
C
      F(1,N,1)=FUNC(1)
      FN(1,N,1)=FUNC(2)
      FNN(1,N,1)=FUNC(3)
      FNNN(N)=DERF(3)
C
      IF (FUNC(2).GT.-5D-6) ISOL(2)=3
      IF (FUNC(3).LT.5D-6) ISOL(2)=1
      IF (ISOL(2).NE.0) PRMT(5)=1D0
C
      IF (IO.EQ.0) GO TO 30
C
      WRITE (6,10) ETA,N,IHLF,(FUNC(I),I=1,3),DERF(3),(ISOL(I),I=1,2)
   10 FORMAT (' '/T5,'ETA=',F8.5,T22,'N=',I3,T32,'IHLF=',I3,T40,4D20.8,I
     16,I6)
      IF (ISOL(2).NE.0) WRITE (6,20)
   20 FORMAT (' '///)
```

```
C
   30  CONTINUE
C
       RETURN
       END
```

```
      SUBROUTINE STATF2(STOREA)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION STOREA(26,71,3),TEMP(3,50,3),HD(6),VD(6)
      DIMENSION FY(71),FX(6),ARGY(6),ARGX(6),VAL(6)
C
      EPS=5D-6
C
      DO 60 K=1,3
C
      D=1.01D0**(DFLOAT(K)/2D0)
C
      DO 60 II=1,3
C
      X=DSQRT(1D0/1.01D0)*DFLOAT(II)/10D0
C
      DO 60 JJ=1,38
C
      Y=DSQRT(1.01D0)*DFLOAT(JJ-1)/10D0
C
      DO 30 I=1,5
C
      DO 10 J=1,71
C
      FY(J)=STOREA(I,J,K)
C
   10 CONTINUE
C
      CALL DATSE(Y,0D0,0.1D0,FY,71,1,ARGY,VAL,6)
      CALL DALI(Y,ARGY,VAL,P,6,EPS,IER)
C
      IJ=I+1
      HD(IJ)=P
C
      IF (IER.NE.0) IER=IER+10
      IF (IER.NE.0) WRITE (6,20) IER
      IF (IER.NE.0) STOP
C
   20 FORMAT (' '/T2,'IER=',I2/)
C
   30 CONTINUE
C
      HD(1)=0D0
C
      DO 50 J=1,6
C
      JARG=10D0*(ARGY(J)+1D-10)+1
C
      DO 40 I=2,6
C
      FX(I)=STOREA(I-1,JARG,K)
C
   40 CONTINUE
C
      FX(1)=0D0
```

```
C
      CALL DATSE(X,0D0,0.1D0,FX,6,1,ARGX,VAL,6)
      CALL DALI(X,ARGX,VAL,P,6,EPS,IER)
C
      VD(J)=P
C
      IF (IER.NE.0) IER=IER+20
      IF (IER.NE.0) WRITE (6,20) IER
      IF (IER.NE.0) STOP
C
   50 CONTINUE
C
      CALL DATSE(X,0D0,0.1D0,HD,6,1,ARGX,VAL,6)
      CALL DALI(X,ARGX,VAL,P1,6,EPS,IER)
C
      IF (IER.NE.0) IER=IER+30
      IF (IER.NE.0) WRITE (6,20) IER
      IF (IER.NE.0) STOP
C
      CALL DALI(Y,ARGY,VD,P2,6,EPS,IER)
C
      IF (IER.NE.0) IER=IER+40
      IF (IER.NE.0) WRITE (6,20) IER
      IF (IER.NE.0) STOP
C
      TEMP(II,JJ,K)=(P1+P2)/(2D0*D)
C
   60 CONTINUE
C
      DO 70 K=1,3
      DO 70 I=1,26
      DO 70 J=1,71
C
      STOREA(I,J,K)=0D0
C
   70 CONTINUE
C
      DO 80 K=1,3
      DO 80 I=1,3
      DO 80 J=1,38
C
      STOREA(I,J,K)=TEMP(I,J,K)
C
   80 CONTINUE
C
      RETURN
      END
```

F.2 The Energy Equation Program

```
C       MAIN PROGRAM
C
        IMPLICIT REAL*8 (A-H,O-Z)
        DIMENSION F(3,71,3),FN(3,71,3),FNN(71)
        DIMENSION S(3,71,3),SN(3,71,3),SNN(71)
        DIMENSION PRMT(5),FUNC(2),DERS(2),AUX(16,2)
        DIMENSION STORF(26,71,3,3),STORAF(26,71,3),STORBF(3,71,3)
        DIMENSION STORS(3,71,2,2),STORAS(26,71,2),STORBS(3,71,2)
        DIMENSION SNS(3),ISOL(2),SSP(71,2)
C
        COMMON S,SN,SNN,F,FN,FNN,ISOL,NE/FCTDAT/EQN,X,DX,T,DT,G
        COMMON /INTDAT/IX,MPS,KT,NS
C
        EXTERNAL FCT,OUTP
C
        IC=0
        IC=1
C
        KT=1
        KT=2
        KT=3
        KT=0
C
        NS=2
        NS=3
C
        UE3=1.01D0
C
        IEX=26
        IEX=3
C
        IBSR=0
        NDIM=2
C
        PRMT(2)=7D0
        PRMT(3)=0.1D0
        PRMT(4)=5D-2
        ERROR=5D-6
C
        IF (KT.EQ.0) ITERS=50
        ITERS=30
        ITERS=50
C
        DO 30 L=1,3
C
        DO 30 J=1,71
C
        DO 30 I=1,26
C
        STORAF(I,J,L)=0D0
        STORF(I,J,1,L)=0D0
        STORF(I,J,2,L)=0D0
        STORF(I,J,3,L)=0D0
        IF (I.GT.3) GO TO 10
        STORBF(I,J,L)=0D0
```

```
C
   10   CONTINUE
C
        IF  (L.EQ.3)  GO TO 20
C
        STORAS(I,J,L)=0D0
        IF  (I.GT.3)  GO TO 20
        STORS(I,J,1,L)=0D0
        STORS(I,J,2,L)=0D0
        STORBS(I,J,L)=0D0
C
   20   CONTINUE
C
   30   CONTINUE
C
        DO 40 J=1,71
C
        FNN(J)=0D0
        SNN(J)=0D0
C
        DO 40 I=1,3
C
        DO 40 K=1,3
C
        F(I,J,K)=0D0
        FN(I,J,K)=0D0
        S(I,J,K)=0D0
        SN(I,J,K)=0D0
C
   40   CONTINUE
C
        IF  (KT.EQ.0)  GO TO 60
C
        READ  (7)  STORAS
C
        DO 50 L=1,2
        DO 50 I=1,3
        DO 50 J=1,71
C
        STORS(I,J,1,L)=STORAS(I,J,L)
C
   50   CONTINUE
C
   60   CONTINUE
C
        IF  (KT.LT.2.OR.KT.GT.3)  GO TO 80
C
        IF  (KT.EQ.2)  READ  (8)  STORBS
        IF  (KT.EQ.3)  READ  (9)  STORBS
C
        DO 70 L=1,2
        DO 70 I=1,3
        DO 70 J=1,71
C
        STORS(I,J,2,L)=STORAS(I,J,L)
```

```
      STORS(I,J,1,L)=STORBS(I,J,L)
C
   70 CONTINUE
C
   80 CONTINUE
C
      IF (KT.EQ.0.AND.NS.EQ.2) READ (1) STORAF
      IF (KT.EQ.0.AND.NS.EQ.3) READ (5) STORAF
      IF (KT.EQ.1) READ (2) STORBF
      IF (KT.EQ.2) READ (3) STORBF
      IF (KT.EQ.3) READ (4) STORBF
C
      IF (KT.GE.2) K=3
      IF (KT.EQ.1) K=2
      IF (KT.EQ.0) K=1
C
      DO 100 L=1,3
      DO 100 I=1,IEX
      DO 100 J=1,71
C
      STORF(I,J,K,L)=STORAF(I,J,L)
      IF (KT.EQ.0) GO TO 90
      STORF(I,J,1,L)=STORBF(I,J,L)
C
   90 CONTINUE
C
  100 CONTINUE
C
      IF (KT.EQ.2) READ (2) STORBF
      IF (KT.EQ.3) READ (3) STORBF
C
      IF (KT.LT.2) GO TO 120
C
      DO 110 L=1,3
      DO 110 I=1,IEX
      DO 110 J=1,71
C
      STORF(I,J,2,L)=STORBF(I,J,L)
C
  110 CONTINUE
C
  120 CONTINUE
C
      DO 130 I=1,3
C
      SNS(I)=0D0
C
  130 CONTINUE
C
      G=5D0/3D0
      G=1.4D0
      TMACCO=1D0
      ENACCO=1D0
C
      DX=0.1D0
```

```
      DN=0.1D0
      DT=0.05D0
C
      SNW=0.2D0
C
      DSNW=0.1D0
C
      PI=3.141592653589793D0
      DFX=DX
      DTEE=DT
      X=0D0
      T=0D0
      C1=(2D0-TMACCO)/TMACCO
      C2=2D0*G/(G+1D0)*(2D0-ENACCO)/ENACCO
      TE=0D0
      TP=1D0
      TP=0.5D0
      NF=71
C
      DT=8D0*DTEE
      IF (KT.GE.12) DT=0.01D0
      IF (KT.LE.9) DT=4D0*DTEE
      IF (KT.LE.5) DT=2D0*DTEE
      IF (KT.LE.2) DT=DTEE
C
      IF (KT.GE.12) T=0.5D0+DFLOAT(KT-11)*DT
      IF (KT.LE.11) T=0.3D0+DFLOAT(KT-9)*DT
      IF (KT.LE.9) T=0.1D0+DFLOAT(KT-5)*DT
      IF (KT.LE.5) T=0.1D0+DFLOAT(KT-2)*DT
      IF (KT.LE.2) T=DFLOAT(KT)*DT
C
      DO 460 IX=1,IEX
C
      PRMT(1)=0D0
      IF (KT.EQ.0) GO TO 150
C
      DO 140 K=2,3
C
      KKK=K-1
C
      DO 140 J=1,71
C
      S(1,J,K)=STORS(IX,J,KKK,1)
      SN(1,J,K)=STORS(IX,J,KKK,2)
C
  140 CONTINUE
C
  150 CONTINUE
C
      DO 160 J=1,71
C
      FNN(J)=STORF(IX,J,1,3)
C
      F(1,J,1)=STORF(IX,J,1,1)
      F(1,J,2)=STORF(IX,J,2,1)
```

```
      F(1,J,3)=STORF(IX,J,3,1)
C
      FN(1,J,1)=STORF(IX,J,1,2)
      FN(1,J,2)=STORF(IX,J,2,2)
      FN(1,J,3)=STORF(IX,J,3,2)
C
      IF (IX.EQ.1) GO TO 160
C
      F(2,J,1)=STORF(IX-1,J,1,1)
      FN(2,J,1)=STORF(IX-1,J,1,2)
C
      IF (IX.EQ.2.OR.IX.EQ.7.OR.IX.EQ.14.OR.IX.EQ.20) GO TO 160
C
      F(3,J,1)=STORF(IX-2,J,1,1)
      FN(3,J,1)=STORF(IX-2,J,1,2)
C
  160 CONTINUE
C
      IF (IX.EQ.1) GO TO 180
C
      IJ=1
C
      IF (IX.EQ.2.OR.IX.EQ.7.OR.IX.EQ.14.OR.IX.EQ.20) IJ=2
C
      DO 170 II=IJ,2
C
      I=4-II
      IJI=I-1
C
      DO 170 J=1,71
C
      S(I,J,1)=S(IJI,J,1)
      SN(I,J,1)=SN(IJI,J,1)
C
  170 CONTINUE
C
  180 CONTINUE
C
      MDPTCR=0
      LF=0
      LF1=0
C
      DX=8D0*DEX
      IF (IX.LE.19) DX=4D0*DEX
      IF (IX.LE.13) DX=2D0*DEX
      IF (IX.LE.6) DX=DEX
      X=X+DX
C
      EQN=2D0
      IF (IX.GT.1.AND.KT.EQ.0) EQN=3D0
      IF (IX.EQ.1.AND.KT.EQ.1.OR.KT.EQ.12) EQN=22D0
      IF (IX.GT.1.AND.KT.EQ.1.OR.KT.EQ.12) EQN=32D0
      IF (IX.EQ.1.AND.KT.GT.1) EQN=23D0
      IF (IX.GT.1.AND.KT.GT.1) EQN=33D0
C
```

```
 190  CONTINUE
C
      IBSR=0
      ISOL(1)=0
C
      MPS=1
      IF (MDPTCR.EQ.1) MPS=11
C
      IF (KT.EQ.0.AND.IX.GT.2) SNW=2D0*SN(2,1,1)-SN(3,1,1)
      IF (KT.EQ.1) SNW=SN(1,1,2)
      IF (KT.GT.1) SNW=2D0*SN(1,1,2)-SN(1,1,3)
      IF (MPS.EQ.11) SNW=SN(1,11,1)
C
      IF (KT.EQ.0.AND.IX.GT.2) DSNW=DABS(SN(2,1,1)-SN(3,1,1))/10D0
      IF (KT.EQ.0.AND.IX.GT.2) DSNW=DABS(SN(2,1,1)-SN(3,1,1))
      IF (KT.EQ.0.AND.IX.GT.2) DSNW=0.01D0
      IF (KT.EQ.1) DSNW=0.01D0*SN(1,1,2)
      IF (KT.GT.1) DSNW=DABS(SN(1,1,2)-SN(1,1,3))/10D0
      IF (MPS.EQ.11) DSNW=0.01D0*SNW
C
      IF (MPS.EQ.11) SNA=SNW
C
 200  CONTINUE
C
      IF (MDPTCR.EQ.0) LF=LF+1
      IF (MDPTCR.EQ.1) LF1=LF1+1
      ISOL(2)=0
C
      DERS(1)=0.3D0
      DERS(2)=0.7D0
C
      IF (MDPTCR.EQ.1) GO TO 210
C
      FUNC(2)=SNW
      FUNC(1)=TP+C2*FUNC(2)/X+(C1**2-2D0*C1*C2)*(FNN(1)/X)**2-1D0
C
      GO TO 220
C
 210  CONTINUE
C
      PRMT(1)=1D0
      FUNC(2)=SNW
      FUNC(1)=S(1,11,1)
C
 220  CONTINUE
C
      SNS(2)=FUNC(2)
C
      CALL DHPCG(PRMT,FUNC,DERS,NDIM,IHLF,FCT,OUTP,AUX)
C
      IF (IHLF.GE.11) WRITE (6,230) IHLF
 230  FORMAT (' '/T10,'IHLF=',I2/)
      IF (IHLF.GE.11) STOP
C
      IF (DABS(FUNC(1)).LT.ERROR) GO TO 290
```

```
      IF (LF1.EQ.50) GO TO 290
      IF (LF.EQ.1.OR.(LF1.EQ.1.AND.MPS.EQ.11)) ISOL(1)=ISOL(2)
      IF (IBSR.EQ.2) GO TO 270
      IF ((LF.NE.1.OR.(LF1.NE.1.AND.MPS.EQ.11)).AND.(ISOL(1).NE.
     1ISOL(2).OR.IBSR.GT.0)) GO TO 240
C
      SNS(ISOL(2))=SNS(2)
      IF (ISOL(2).EQ.1) SNW=SNW+DSNW
      IF (ISOL(2).EQ.3) SNW=SNW-DSNW
C
      GO TO 200
C
  240 CONTINUE
C
      IBSR=1
C
      SNS(ISOL(2))=SNS(2)
      IF (DABS(FUNC(1)).LT.5D-3) GO TO 250
      SNW=(SNS(1)+SNS(3))/2D0
      IF (LF+LF1.EQ.ITERS) MDPTCR=1
      IF (LF+LF1.EQ.ITERS) GO TO 190
C
      GO TO 200
C
  250 CONTINUE
C
      IBSR=2
      SP=FUNC(1)
      SNP=SNW
C
      SNW=SNS(1)
      IF (ISOL(2).EQ.1) SNW=SNS(3)
C
      DO 260 J=1,71
C
      SSP(J,1)=S(1,J,1)
      SSP(J,2)=SN(1,J,1)
C
  260 CONTINUE
C
      GO TO 200
C
  270 CONTINUE
C
      SA=-FUNC(1)/(SP-FUNC(1))
      SB=1D0-SA
C
      DO 280 J=1,71
C
      S(1,J,1)=SA*SSP(J,1)+SB*S(1,J,1)
      SN(1,J,1)=SA*SSP(J,2)+SB*SN(1,J,1)
C
  280 CONTINUE
C
  290 CONTINUE
```

```
C
      IF (NE.EQ.71) GO TO 310
C
      JS=NE+1
      DO 300 J=JS,71
C
      DEL=1D0-DFLOAT(71-J)/DFLOAT(71-NE)
      S(1,J,1)=DEL*S(1,J,1)
      SN(1,J,1)=DEL*SN(1,J,1)
      SNN(J)=DEL*SNN(J)
C
 300  CONTINUE
C
 310  CONTINUE
C
      IF (MPS.EQ.1) GO TO 330
C
      DELSN=(SN(1,11,1)-SNA)/2D0
C
      DO 320 J=2,20
C
      ETA=DFLOAT(J-1)/10D0
      IF (J.LE.11) SN(1,J,1)=SN(1,J,1)+ETA*DELSN
      IF (J.GT.11) SN(1,J,1)=SN(1,J,1)-(2D0-ETA)*DELSN
C
 320  CONTINUE
C
 330  CONTINUE
C
      DO 340 J=1,NE
C
      S(1,J,1)=S(1,J,1)+1D0
C
 340  CONTINUE
C
      FSETA=DN*DFLOAT(NE-1)
C
      IF (IX.EQ.1) WRITE (6,350)
 350  FORMAT ('1')
C
      TS=T
      IF (KT.EQ.0.AND.NS.EQ.3) T=10D0
      WRITE (6,360) X,FSETA,T
      T=TS
 360  FORMAT ('0',T10,'X=',F4.1,T20,'ETA(FREE STREAM)=',F3.1,
     1T45,'T=',F6.4)
C
      WRITE (6,370)
 370  FORMAT (' '///T10,'UNSTEADY STATE (WITH SLTP) ENTHALPY DISTRIBUTIO
     1N:'/)
C
      WRITE (6,380) (S(1,J,1),J=1,NE)
 380  FORMAT (' ',T2,5F23.5)
C
      WRITE (6,390)
```

```
 390   FORMAT (' '///T10,'UNSTEADY STATE (WITH SLIP) S-PRIME DISTRIBUTION
      1:'/)
 C
       WRITE (6,380) (SN(1,J,1),J=1,NE)
 C
       LFS=LF+LF1
 C
       WRITE (6,400) LF,LF1,LFS
 400   FORMAT (' '/////T10,'CONVERGED IN ',I2,'+',I2,'=',I2,' ITERATIONS.
      1')
 C
       WRITE (6,350)
 C
       DO 410 J=1,NE
 C
       S(1,J,1)=S(1,J,1)-1D0
 C
 410   CONTINUE
 C
       IF (KT.EQ.0) GO TO 430
 C
       JI=2
       IF (KT.EQ.0.OR.KT.EQ.2.OR.KT.EQ.5.OR.KT.EQ.9.OR.KT.EQ.11) JI=1
 C
       DO 420 K=1,JI
 C
       DO 420 J=1,71
 C
       STORS(IX,J,K,1)=S(1,J,K)
       STORS(IX,J,K,2)=SN(1,J,K)
 C
 420   CONTINUE
 C
 430   CONTINUE
 C
       IF (KT.GT.0) GO TO 450
 C
       DO 440 J=1,71
 C
       STORAS(IX,J,1)=S(1,J,1)
       STORAS(IX,J,2)=SN(1,J,1)
 C
 440   CONTINUE
 C
 450   CONTINUE
 C
 460   CONTINUE
 C
       IF (IO.EQ.0) GO TO 490
       IF (KT.NE.0) GO TO 470
 C
       IF (NS.EQ.2) WRITE (7) STORAS
       IF (NS.EQ.3) WRITE (11) STORAS
 C
       GO TO 490
```

```
C
 470   CONTINUE
C
       DO 480 I=1,3
       DO 480 L=1,2
       DO 480 J=1,71
C
       STORBS(I,J,L)=STORS(I,J,1,L)
C
 480   CONTINUE
C
       IF (KT.EQ.1) WRITE (8) STORBS
       IF (KT.EQ.2) WRITE (9) STORBS
       IF (KT.EQ.3) WRITE (10) STORBS
C
 490   CONTINUE
C
       STOP
       END
```

```
      SUBROUTINE FCT(ETA,FUNC,DERS)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION F(3,71,3),FN(3,71,3),FNN(71)
      DIMENSION S(3,71,3),SN(3,71,3),SNN(71)
      DIMENSION FT(61,2),ARG(20),VAL(40)
      DIMENSION FUNC(2),DERS(2)
C
      COMMON S,SN,SNN,F,FN,FNN
      COMMON /FCTDAT/FQN,X,DX,T,DT,G
      COMMON /INTDAT/IX,MPS,KT,NS
C
      ERR=5D-6
C
      N1=MPS+60
      N2=20
      N3=61
      EPS=ERR
C
      BGN=0D0
      IF (MPS.EQ.11) BGN=1D0
C
      DERS(1)=FUNC(2)
C
      INTERP=1
C
      DO 10 J=MPS,71
C
      N=J
C
      ETACHK=0.1D0*DFLOAT(N-1)
      DETA=ETA-ETACHK
C
      IF (DABS(DETA).LE.5D-10) INTERP=0
      IF (INTERP.EQ.0) ETA=ETACHK
      IF (INTERP.EQ.0) GO TO 20
      IF (DETA.LE.0D0) GO TO 20
C
   10 CONTINUE
C
   20 CONTINUE
C
      FAXT=F(1,N,1)
      FNAXT=FN(1,N,1)
C
      FM1X=F(2,N,1)
      FM2X=F(3,N,1)
C
      FM1T=F(1,N,2)
      FM2T=F(1,N,3)
C
      SM1X=S(2,N,1)
      SM2X=S(3,N,1)
C
      SM1T=S(1,N,2)
```

```
      SM2T=S(1,N,3)
C
      IF (INTERP.EQ.0) GO TO 110
C
      IK=0
      DO 30 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=F(1,J,1)
      FT(JJ,2)=FN(1,J,1)
C
   30 CONTINUE
C
      IK=IK+1
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
      FAXT=P
C
      IF (IER.GT.1) WRITE (6,40) IER
   40 FORMAT (' '/T10,'IER=',I1,' FOR FAXT.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 30
      IF (IER.EQ.1) WRITE (6,40) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
      IK=0
      DO 50 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=FN(1,J,1)
      FT(JJ,2)=FNN(J)
C
   50 CONTINUE
C
      IK=IK+1
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
      FNAXT=P
C
      IF (IER.GT.1) WRITE (6,60) IER
   60 FORMAT (' '/T10,'IER=',I1,' FOR FNAXT.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 50
      IF (IER.EQ.1) WRITE (6,60) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
      IK=0
      DO 70 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=F(2,J,1)
      FT(JJ,2)=FN(2,J,1)
C
```

```
   70 CONTINUE
C
      IK=IK+1
      CALL DATSL(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
      FM1X=P
C
      IF (IER.GT.1) WRITE (6,80) IER
   80 FORMAT (' '/T10,'IER=',I1,' FOR FM1X.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 70
      IF (IER.EQ.1) WRITE (6,80) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
      IK=0
      DO 90 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=S(2,J,1)
      FT(JJ,2)=SN(2,J,1)
C
   90 CONTINUE
C
      IK=IK+1
      CALL DATSL(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
      SM1X=P
C
      IF (IER.GT.1) WRITE (6,100) IER
  100 FORMAT (' '/T10,'IER=',I1,' FOR SM1X.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 90
      IF (IER.EQ.1) WRITE (6,100) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
  110 CONTINUE
C
      IF (EQN.NE.2D0) GO TO 120
C
      DERS(2)=-(FAXT+ETA)*FUNC(2)-X*((FAXT-FM1X)*FUNC(2)-(FUNC(1)-SM1X)*
     1(FNAXT+1D0))/DX
C
      RETURN
C
  120 CONTINUE
C
      IF (INTERF.EQ.0.OR.EQN.EQ.3D0) GO TO 170
C
      IK=0
      DO 130 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=F(1,J,2)
```

```
      FT(JJ,2)=FN(1,J,2)
C
  130 CONTINUE
C
      IK=IK+1
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
      FM1T=P
C
      IF (IER.GT.1) WRITE (6,140) IER
  140 FORMAT (' '/T10,'IER=',I1,' FOR FM1T.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 130
      IF (IER.EQ.1) WRITE (6,140) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
      IK=0
      DO 150 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=S(1,J,2)
      FT(JJ,2)=SN(1,J,2)
C
  150 CONTINUE
C
      IK=IK+1
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
      SM1T=P
C
      IF (IER.GT.1) WRITE (6,160) IER
  160 FORMAT (' '/T10,'IER=',I1,' FOR SM1T.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 150
      IF (IER.EQ.1) WRITE (6,160) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
  170 CONTINUE
C
      IF (EQN.NE.22D0) GO TO 180
C
      DERS(2)=-(FAXT+ETA)*FUNC(2)-X*((FAXT-FM1X)*FUNC(2)-(FUNC(1)-SM1X)*
     1(FNAXT+1D0))/DX+2D0*(1D0-T)**2*(FUNC(1)-SM1T)/DT-2D0*T*(1D0-T)*((F
     2UNC(1)-SM1T)*(FNAXT+1D0)-(FAXT-FM1T)*FUNC(2))/DT
C
      RETURN
C
  180 CONTINUE
C
      IF (INTERP.EQ.0) GO TO 230
C
      IK=0
      DO 190 J=MPS,N1
```

```
C
      JJ=J-MPS+1
      FT(JJ,1)=F(3,J,1)
      FT(JJ,2)=FN(3,J,1)
C
 190  CONTINUE
C
      IK=IK+1
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
      FM2X=P
C
      IF (IER.GT.1) WRITE (6,200) IER
 200  FORMAT (' '/T10,'IER=',I1,' FOR FM2X.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 190
      IF (IER.EQ.1) WRITE (6,200) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
      IK=0
      DO 210 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=S(3,J,1)
      FT(JJ,2)=SN(3,J,1)
C
 210  CONTINUE
C
      IK=IK+1
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
      SM2X=P
C
      IF (IER.GT.1) WRITE (6,220) IER
 220  FORMAT (' '/T10,'IER=',I1,' FOR SM2X.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 210
      IF (IER.EQ.1) WRITE (6,220) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
 230  CONTINUE
C
      IF (EQN.NE.3D0) GO TO 240
C
      DFRS(2)=-(FAXT+ETA)*FUNC(2)-X*((1.5D0*FAXT-2D0*FM1X+0.5D0*FM2X)*FU
     1NC(2)-(1.5D0*FUNC(1)-2D0*SM1X+0.5D0*SM2X)*(FNAXT+1D0))/DX
C
      RETURN
C
 240  CONTINUE
C
      IF (EQN.NE.32D0) GO TO 250
C
```

```
      DFRS(2)=-(FAXT+ETA)*FUNC(2)-X*((1.5D0*FAXT-2D0*FM1X+0.5D0*FM2X)*FU
     1NC(2)-(1.5D0*FUNC(1)-2D0*SM1X+0.5D0*SM2X)*(FNAXT+1D0))/DX+2D0*(1D0
     2-T)**2*(FUNC(1)-SM1T)/DT-2D0*T*(1D0-T)*((FUNC(1)-SM1T)*(FNAXT+1D0)
     3-(FAXT-FM1T)*FUNC(2))/DT
C
      RETURN
C
  250 CONTINUE
C
      IF (INTERP.EQ.0) GO TO 300
C
      IK=0
      DO 260 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=F(1,J,3)
      FT(JJ,2)=FN(1,J,3)
C
  260 CONTINUE
C
      IK=IK+1
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,P,N2,EPS,IER)
      FM2T=P
C
      IF (IER.GT.1) WRITE (6,270) IER
  270 FORMAT (' '/T10,'IER=',I1,' FOR FM2T.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 260
      IF (IER.EQ.1) WRITE (6,270) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
      IK=0
      DO 280 J=MPS,N1
C
      JJ=J-MPS+1
      FT(JJ,1)=S(1,J,3)
      FT(JJ,2)=SN(1,J,3)
C
  280 CONTINUE
C
      IK=IK+1
      CALL DATSE(ETA,BGN,0.1D0,FT,N3,2,ARG,VAL,N2)
      CALL DAHI(ETA,ARG,VAL,1,N2,EPS,IER)
      SM2T=P
C
      IF (IER.GT.1) WRITE (6,290) IER
  290 FORMAT (' '/T10,'IER=',I1,' FOR SM2T.'/)
      IF (IER.EQ.1) EPS=10D0*EPS
      IF (IER.EQ.1.AND.IK.LT.5) GO TO 280
      IF (IER.EQ.1) WRITE (6,290) IER
      IF (IER.GT.0) STOP
      EPS=ERR
C
```

```
 300  CONTINUE
C
      IF (EQN.NE.23D0) GO TO 310
C
      DERS(2)=-(FAXI+ETA)*FUNC(2)-X*((FAXT-FM1X)*FUNC(2)-(FUNC(1)-SM1X)*
     1(FNAXT+1D0))/DX+2D0*(1D0-T)**2*(1.5D0*FUNC(1)-2D0*SM1T+0.5D0*SM2T)
     2/DT-2D0*T*(1D0-T)*((1.5D0*FUNC(1)-2D0*SM1T+0.5D0*SM2T)*(FNAXT+1D0)
     3-(1.5D0*FAXT-2D0*FM1T+0.5D0*FM2T)*FUNC(2))/DT
C
      RETURN
C
 310  CONTINUE
C
      DERS(2)=-(FAXI+ETA)*FUNC(2)-X*((1.5D0*FAXT-2D0*FM1X+0.5D0*FM2X)*FU
     1NC(2)-(1.5D0*FUNC(1)-2D0*SM1X+0.5D0*SM2X)*(FNAXT+1D0))/DX+2D0*(1D0
     2-T)**2*(1.5D0*FUNC(1)-2D0*SM1T+0.5D0*SM2T)/DT-2D0*T*(1D0-T)*((1.5D
     30*FUNC(1)-2D0*SM1T+0.5D0*SM2T)*(FNAXT+1D0)-(1.5D0*FAXT-2D0*FM1T+0.
     45D0*FM2T)*FUNC(2))/DT
C
      RETURN
      END
```

```
      SUBROUTINE OUTP(ETA,FUNC,DERS,IHLF,NDIM,PRMT)
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION F(3,71,3),FN(3,71,3),FNN(71)
      DIMENSION S(3,71,3),SN(3,71,3),SNN(71)
      DIMENSION PRMT(5),FUNC(2),DERS(2),ISOL(2)
C
      COMMON S,SN,SNN,F,FN,FNN,ISOL,N
      COMMON /INTDAT/IX,MPS,KT,NS
C
      IF (KT.NE.0) GO TO 2
C
      IF (IX.EQ.1) NE=42
      IF (IX.EQ.2) NE=44
      IF (IX.EQ.3) NE=45
      IF (NS.EQ.2) NE=39
C
    2 CONTINUE
C
      IF (KT.NE.1) GO TO 4
C
      IF (IX.EQ.1) NE=40
      IF (IX.EQ.2) NE=41
      IF (IX.EQ.3) NE=42
C
    4 CONTINUE
C
      IF (KT.NE.2) GO TO 6
C
      IF (IX.EQ.1) NE=41
      IF (IX.EQ.2) NE=42
      IF (IX.EQ.3) NE=42
C
    6 CONTINUE
C
      IF (KT.NE.3) GO TO 7
C
      IF (IX.EQ.1) NE=42
      IF (IX.EQ.2) NE=43
      IF (IX.EQ.3) NE=44
C
    7 CONTINUE
C
      NX=0
      NX=3
      NY=2
      NX=1
      NX=4
      NX=5
C
      INTERP=1
C
      DO 10 J=1,71
C
      N=J
```

```
C
      ETACHK=0.1D0*DFLOAT(N-1)
      DETA=ETA-ETACHK
C
      IF (DABS(DETA).LE.5D-10) INTERP=0
      IF (INTERP.EQ.0) ETA=ETACHK
      IF (INTERP.EQ.0) GO TO 20
      IF (DETA.LE.0D0) GO TO 20
C
   10 CONTINUE
C
   20 CONTINUE
C
      IO=1
      IF (IX.GT.NX) IO=1
      IO=0
C
      IF (INTERP.NE.0) RETURN
C
      S(1,N,1)=FUNC(1)
      SN(1,N,1)=FUNC(2)
      SNN(N)=DERS(2)
C
      IF (FUNC(1).GT.0D0) ISOL(2)=3
      IF (FUNC(1).LE.0D0) ISOL(2)=1
      IF (DABS(DERS(2)).GT.1D2) PRMT(5)=1D0
      IF (N.EQ.NE) PRMT(5)=1D0
C
      IF (IO.EQ.0) GO TO 50
C
      WRITE (6,30) ETA,N,IHLF,(FUNC(I),I=1,2),DERS(2),(ISOL(I),I=1,2)
   30 FORMAT (' '/T5,'ETA=',F8.5,T22,'N=',I3,T32,'IHLF=',I3,T40,3D20.8,I
     15,I6)
C
      IF (N.EQ.NE) WRITE (6,40)
   40 FORMAT (' '///)
C
   50 CONTINUE
C
      RETURN
      END
```